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INDIA-TAIWAN CONFERENCE ON DISCRETE MATHEMATICS

National Taiwan University
November 9 to 12, 2009

EXTENDED ABSTRACTS

National Center for Theoretical Sciences

India-Taiwan Conference on Discrete Mathematics
National Taiwan University, Taipei 10617, Taiwan
November 9 to 12, 2009

This is the first joint conference in Discrete Mathematics between India and Taiwan. The aim of this conference is to provide a platform for researchers in discrete mathematics in India and Taiwan to share results, problems and ideas. To provide an opportunity for mathematicians from both sides to have better understanding of each other's research, and hopefully to initiate collaborations among them.

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Contents

Component-complete sets in graphs

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Abstract

Let $G = (V, E)$ be a graph. A subset S of V is called a component-complete set or *cc*-set if every component of the induced subgraph $\langle S \rangle$ is complete. In this paper we introduce several parameters using *cc*-sets and discuss their relations with other graph theoretic parameters.

Keywords : Domination, irredundance, component-complete sets, *cc*-domination, *cc*-irredundance.

2000 Mathematics Subject Classification: 05C69

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, irredundance, covering and matching. An excellent treatment of fundamentals of domination in graphs is given in the book by Haynes et al. [2]. Surveys of several advanced topics in domination are given in the book edited by Haynes et al. [6].

Let $G = (V, E)$ be a graph. Let $v \in V$. The open neighbourhood $N(v)$ and closed neighbourhood $N[v]$ are defined by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of V is said to be an independent set if no two vertices in S are adjacent. A set S is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. A subset S of V is called an irredundant set if for every vertex $v \in S$ there exists a vertex w such that $N[w] \cap S = \{v\}$.

We observe that the maximality condition for an independent set is the definition of dominating set and the minimality condition for a dominating set is the definition of irredundant set.

The above concepts of independence, domination and irredundance lead to the following six parameters.

$$i(G) = min\{|S| : S \text{ is a maximal independent set in } G\}.$$
$$\beta_0(G) = max\{|S| : S \text{ is an independent set in } G\}.$$
$$\gamma(G) = min\{|S| : S \text{ is a dominating set in } G\}.$$
$$\Gamma(G) = max\{|S| : S \text{ is a minimal dominating set in } G\}.$$
$$ir(G) = min\{|S| : S \text{ is a maximal irredundant set in } G\}.$$
$$IR(G) = max\{|S| : S \text{ is an irredundant set in } G\}.$$

These parameters are respectively called independent domination number, independence number, domination number, upper domination number, irredundance number and upper irredundance number. These parameters satisfy the following inequality chain.

$$ir(G) \le \gamma(G) \le i(G) \le \beta_0(G) \le \Gamma(G) \le IR(G).$$

This inequality chain which was first observed by Cockayne et al. [1] is called the domination chain and has become one of the strongest focal points for research in domination theory.

There are many variations of domination in graphs. In the book [2] it is proposed that a type of domination is "fundamental" if every connected nontrivial graph has a dominating set of this type and this type of dominating set S is defined in terms of some "natural" property of the subgraph induced by S. Examples include total domination, independent domination, connected domination and paired domination. In this paper we introduce the concept of component-complete sets, which is a fundamental concept in the above sense. We also introduce several new parameters using this concept and investigate their relation with the six basic parameters of the domination chain.

We need the following definitions.

Definition 1 *The cartesian product* $G = G_1 \Box G_2$ *has* $V(G) = V(G_1) \times V(G_2)$, *and two vertices* (u_1, u_2) *and* (v_1, v_2) *of* G *are adjacent if and only if either* $u_1 = v_1$ *and* $u_2v_2 \in E(G_2)$ *or* $u_2 = v_2$ *and* $u_1v_1 \in E(G_1)$.

Definition 2 *The corona of two graphs* G_1 *and* G_2 *is the graph* $G = G_1 \circ G_2$ *formed from one copy of* G_1 *and* $|V(G_1)|$ *copies of* G_2 *where the* i^{th} *vertex of* G_1 *is adjacent to every vertex in the* i^{th} *copy of* G_2.

2 Basic Results

Dutton and Brigham [4] have proved the following theorem for claw-free graphs.

Theorem 3 [4] *Let D be a minimal dominating set of vertices in a $K_{1,3}$-free graph. Then D is a collection of disjoint complete subgraphs.*

Hence if D is a minimal dominating set of a claw-free graph G, then every component of the induced subgraph $\langle D\rangle$ is complete. We observe that any independent set S in a graph also has this property. Further the concept of subcoloring and subchromatic number [9, 1, 6] deals with the problem of partitioning vertex set V into subsets $X_1, X_2, \ldots, X_k$, where each X_i is such that every component of $\langle X_i\rangle$ is complete. These observations motivate the concept of component-complete sets.

Definition 4 *Let $G = (V, E)$ be a graph. A subset S of V is called a component-complete set or simply a cc-set if every component of the induced subgraph $\langle S\rangle$ is complete.*

The concept of component-completness is a natural generalization of the concept of independence, since every independent set is obviously a *cc*-set. Also any subset of a *cc*-set is a *cc*-set, so that component-completeness is a hereditary property. Hence a *cc*−set S is a maximal *cc*-set if and only if $S \cup \{v\}$ is not a *cc*−set for all $v \in V - S$. Thus a *cc*-set $S \subseteq V$ is maximal if and only if for every $v \in V - S$, there exists a clique C in $\langle S\rangle$ such that v is adjacent to a vertex in C and v is not adjacent to a vertex in C or there exist two cliques C_1 and C_2 in $\langle S\rangle$ such that v is adjacent to a vertex in C_1 and to a vertex in C_2.

Definition 5 *The cc-number $\beta_{cc}(G)$ and the lower cc-number $i_{cc}(G)$ are defined by*

$$\beta_{cc}(G) = max\{|S| : S \text{ is a maximal cc-set of } G\} \text{ and}$$

$$i_{cc}(G) = min\{|S| : S \text{ is a maximal cc-set of } G\}.$$

Clearly $i_{cc}(G) \leq \beta_{cc}(G)$. Further since every independent set is a *cc*-set, it follows that $\beta_0(G) \leq \beta_{cc}(G)$.

Definition 6 *A dominating set of G which is also a cc-set is called a cc-dominating set of G. The cc-domination number $\gamma_{cc}(G)$ and the upper cc-domination number $\Gamma_{cc}(G)$ are defined by*

$$\gamma_{cc}(G) = min\{|S| : S \text{ is a minimal cc-dominating set of } G\} \text{ and}$$

$$\Gamma_{cc}(G) = max\{|S| : S \text{ is a minimal cc-dominating set of } G\}.$$

Since any maximal cc-set is a dominating set of G, and every maximal independent set is a minimal $cc-$dominating set, the parameters $\gamma_{cc}(G)$ and $\Gamma_{cc}(G)$ fit into the domination chain,thus leading to the following extended domination chain

$$ir(G) \leq \gamma(G) \leq \gamma_{cc}(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_{cc}(G) \leq \Gamma(G) \leq IR(G).$$

We observe that the parameters $i_{cc}(G)$ and $\beta_{cc}(G)$ do not fit into this chain, since $i_{cc}(K_n) = \beta_{cc}(K_n) = n$ and all the other parameters in the chain are equal to 1 for K_n.

Definition 7 *An irredundant set in G which is also cc$-$set is called a cc-irredundant set.*

Observation 8 *Since cc-irredundance is a hereditary property, it follows that a cc-irredundant set S is maximal if and only if for every $x \in V - S$, $S \cup \{x\}$ is either not a cc-set or it is not an irredundant set. In particular, if a cc-set is a maximal irredundant set, then it is necessarily a maximal cc-irredundant set. However, the converse is not true. For the graph G given in Figure 1, the set $I = \{a, b\}$ is a maximal cc-irredundant set, but is not a maximal irredundant set.*

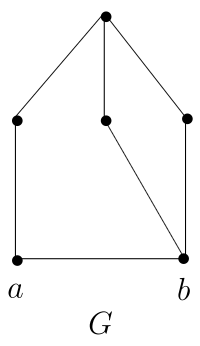

Figure 1

Lemma 9 *Any minimal cc-dominating set is a maximal cc-irredundant set.*

Proof. Let S be a minimal cc-dominating set of G. Since S is a minimal dominating set, S is a maximal irredundant set. Thus, S is a maximal cc-irredundant set of G. ∎

Definition 10 *The cc-irredundance number and the upper cc-irredundance number of G are defined by*

$ir_{cc}(G) = min\{|I| : I \text{ is a maximal cc-irredundant set}\}$ *and*

$IR_{cc}(G) = max\{|I| : I \text{ is a maximal cc-irredundant set}\}.$

For any graph G, we have $ir_{cc}(G) \leq \gamma_{cc}(G) \leq \Gamma_{cc}(G) \leq IR_{cc}(G)$ and $ir_{cc}(G) \leq \gamma_{cc}(G) \leq i_{cc}(G) \leq \beta_{cc}(G)$.

It follows from Theorem 3 that for any claw-free graph G, $\gamma(G) = \gamma_{cc}(G)$ and $\Gamma(G) = \Gamma_{cc}(G)$. Also for any graph G with $\gamma(G) = 2$,we have $\gamma(G) = \gamma_{cc}(G) = 2$. The following theorem gives another family of graphs for which $\gamma(G) = \gamma_{cc}(G)$ and $\Gamma(G) = \Gamma_{cc}(G)$.

Theorem 11 *If G is 3-regular, then $\gamma(G) = \gamma_{cc}(G)$ and $\Gamma(G) = \Gamma_{cc}(G)$.*

Proof. Let G be a 3-regular graph and let D be a $\gamma-$set of G such that $\langle D\rangle$ contains maximum number of components. If $\langle D\rangle$ contains $P_3 = (u, v, w)$ as an induced subgraph, then $D' = (D - \{v\}) \cup \{v_1\}$, where v_1 is a private neighbor of v with respect to D, is a $\gamma-$set and the number of components in $\langle D'\rangle$ is larger than the number of components in $\langle D\rangle$, which is a contradiction. Hence it follows that D is a $cc-$set and $\gamma(G) = \gamma_{cc}(G)$. Similarly we can prove that $\Gamma(G) = \Gamma_{cc}(G)$. ∎

3 Relationship between parameters

In this section we discuss the relation between the standard domination chain parameters $ir, \gamma, i, \beta_0, \Gamma, IR$ and the corresponding cc-parameters $ir_{cc}, \gamma_{cc}, i_{cc}, \beta_{cc}, \Gamma_{cc}$ and IR_{cc}. We first present several pairs of these parameters which are not comparable.

For the graph H given in Figure 2, $ir_{cc}(H) = 4$ and $\gamma(H) = 5$. For the graph G given in Figure 3, $S_1 = \{a, b, c\}$ is a maximal cc-irredundant set of minimum cardinality, so that $ir_{cc}(G) = 3$. However, $S_2 = \{a, e, f, g\}$ is a maximal irredundant set of minimum cardinality, so that $ir(G) = 4$. Also for the graph $G = P_3 \circ 2K_1$, $ir(G) = 3$, $ir_{cc}(G) = 4$ and $\gamma(G) = 3$. Hence the parameters ir and ir_{cc} and the parameters γ and ir_{cc} are not comparable.

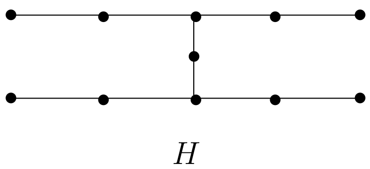

Figure 2

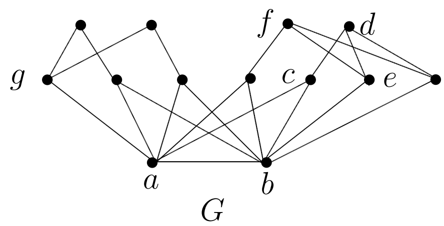

Figure 3

For the complete bipartite graph $G = K_{2,r}, r \geq 3$, we have $i_{cc} = 2$ and $\beta_0(G) = \Gamma_{cc}(G) = \Gamma(G) = IR_{cc}(G) = \beta_{cc}(G) = r$. Also $i_{cc}(K_n) = \beta_{cc}(K_n) = n$ whereas $i(K_n) = \beta_0(K_n) = \Gamma_{cc}(K_n) = \Gamma(K_n) = IR_{cc}(K_n) = IR(K_n) = 1$. Further $i(K_n \circ 2K_1) = 2n - 1$ and $i_{cc}(K_n \circ 2K_1) = n$. Hence i_{cc} is not comparable with any of

$IR, IR_{cc}, \Gamma, \Gamma_{cc}, i(G)$ and β_0. For the graph G obtained from $K_{4,4,4} \circ K_1$, by adding edges in such a way that the subgraph induced by the set of all pendant vertices is a cycle, we have $\Gamma(G) = IR(G) = 12$ and $\beta_{cc}(G) = 11$. Thus β_{cc} is not comparable with IR and Γ.

Let G be the graph obtained from the path $P_6 = (a_1, a_2, a_3, a_4, a_5, a_6)$ and the complete graph K_6 with $V(K_6) = \{b_1, b_2, b_3, b_4, b_5, b_6\}$ by adding the edges $a_1b_1, a_2b_2, a_4b_4, a_5b_5$ and a_6b_6. At least one vertex of $V(K_6)$ is in any dominating set of G, and $\Gamma(G) = 4$. Also $I = \{b_1, b_2, b_4, b_5, b_6\}$ is a cc-irredundant set and so $IR_{cc}(G) > \Gamma(G)$. Also for the graph $H = C_5 \Box K_2$, we have $\Gamma(H) = 5 > 4 = IR_{cc}(H)$. Thus Γ and IR_{cc} are not comparable.

Theorem 12 *Given any positive integer k, there exist graphs G_1, G_2, G_3, G_4, G_5 and G_6 such that $\gamma_{cc}(G_1) - \gamma(G_1) > k$, $i(G_2) - \gamma_{cc}(G_2) > k$, $i_{cc}(G_3) - \gamma_{cc}(G_3) > k$, $\Gamma_{cc}(G_4) - \beta_0(G_4) > k$, $\Gamma(G_5) - \Gamma_{cc}(G_5) > k$ and $ir(G_6) - ir_{cc}(G_6)$.*

Proof. For the graph $G_1 = K_{n,n,n} \circ mK_1, m \geq 2, n \geq 2$, we have $\gamma(G_1) = 3n$ and $\gamma_{cc}(G_1) = n + 2mn$. For the graph G_2 formed from the path (u, v, w) of order 3, by attaching $m, 2m$ and $3m$ pendant vertices at u, v and w respectively, we have $\gamma(G_2) = 3, \gamma_{cc}(G_2) = 2 + m$ and $i(G_2) = 2 + 2m$. Also for the graph $G_3 = K_n$, we have $i_{cc}(G_3) = n$ and $\gamma_{cc}(G_3) = 1$. Hence the differences $\gamma_{cc} - \gamma$, $i - \gamma_{cc}$ and $i_{cc} - \gamma_{cc}$ can be made arbitrarily large. For the cartesian product $G_4 = K_m \Box K_2$, we have $\Gamma_{cc}(G_4) = m$ and $\beta_0(G_4) = 2$. For the graph G_5 obtained from $K_{n,n,n} \circ K_1, n \geq 2$, by adding edges in such a way that the subgraph induced by the set of all pendant vertices is a cycle, we have $\Gamma_{cc}(G_5) = 2n$ and $\Gamma(G_5) = 3n$. Thus the differences $\Gamma_{cc} - \beta_0$ and $\Gamma - \Gamma_{cc}$ can be made arbitrarily large. For the graph G_6 formed from the union of k vertex disjoint copies $H_1, H_2, \ldots, H_k$ of G given in Figure 3 with additional edges so that the vertices $d_1, d_2, \ldots, d_k$, where d_i is the vertex labelled d in the i^{th} copy H_i induce a path. The resulting graph has $ir(G_6) = 4k$ and $ir_{cc}(G_6) = 3k$ and hence $ir - ir_{cc}$ can be made arbitrarily large. ■

Theorem 13 *For any graph $G, \gamma(G) \leq 2ir_{cc}(G)$.*

Proof. Let $I = \{x_1, x_2, \ldots, x_k\}$ be an ir_{cc}-set of G. Let y_i be a private neighbor of x_i with respect to I and let $A = I \cup \{y_1, y_2, \ldots, y_k\}$. If there exists a vertex x in $V - A$ such that $N(x) \cap (V - A) = \emptyset$, then $B = I \cup \{x\}$ is a cc-set of G and x is an isolated vertex in $\langle B \rangle$. Further for each i, y_i is a private neighbor of x_i with respect to B and hence B is cc-irredundant, which is a contradiction. Hence A is a dominating set of G and hence $\gamma(G) \leq 2ir_{cc}(G)$. ■

Observation 14 *For the graph G given in Figure 4, $I = \{a, b, c, d, e, f\}$ is a maximal cc-irredundant set and $ir_{cc}(G) = 6$. However $\gamma_{cc}(G) = 5 + 2t, t \geq 2$ and hence the ratio $\gamma_{cc}(G)/ir_{cc}(G)$ can be arbitrarily large.*

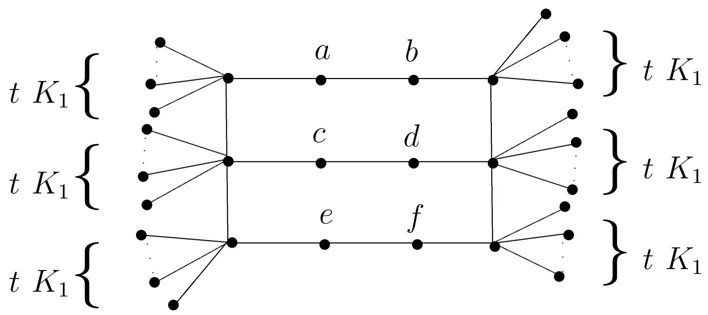

Figure 4

The following Hasse diagram summarizes the relationship between the various parameters for an arbitrary graph G.

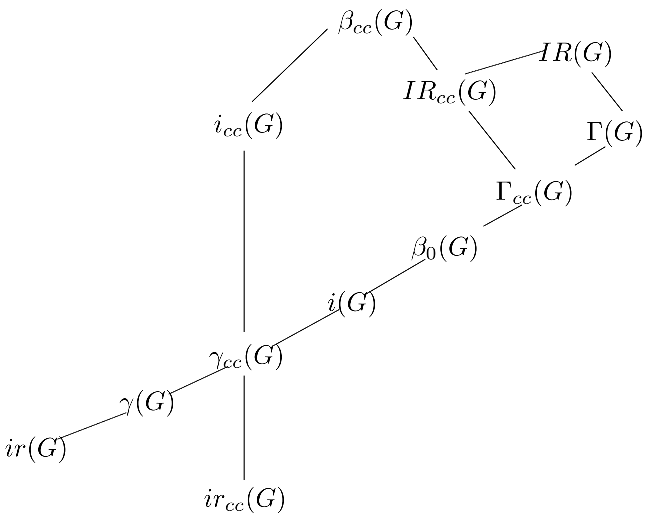

Figure 5. Relationship between parameters

Theorem 15 *Let a, b, c, d be four integers such that $3 \leq a \leq b \leq c \leq d$ and $b \leq 2a-1$. Then there exists a graph G with $ir(G) = a, \gamma(G) = b, \gamma_{cc}(G) = c$ and $i(G) = d$.*

Proof. Case i. $b > a$.

We construct a graph G as follows. The vertex set of G consists of disjoint subsets $A, W_1, W_2, \ldots, W_{b-a}, Y_1, Y_2, \ldots, Y_a$ and $Z_1, Z_2, \ldots, Z_a$, where $A = \{w_1, w_2, \ldots, w_a\}$, $W_j = \{w_{jk} : k = 1, 2, \ldots, a-j\}, j = 1, 2, \ldots, b-a$, $Y_j = \{y_{j1}\}, j = 1, 2, \ldots, a$, $Z_1 = \{z_{11}, z_{12}, \ldots, z_{1d}\}$, $Z_2 = \{z_{21}, z_{22}, \ldots, z_{2m}\}$, $m = d - b + 1$, $Z_3 = \{z_{31}, z_{32}, \ldots, z_{3n}\}, n = c - b + 1$. Also let $Z_j = \{z_{j1}\}, j = 4, 5, \ldots, a$, $W = \bigcup_{i=1}^{b-a} W_i$, $Y = \bigcup_{j=1}^{a} Y_j$ and $Z = \bigcup_{j=1}^{a} Z_j$. Join the vertices of G in such a way that $G[A] \cong K_a$ and $G[W_i] \cong K_{a-j}, j = 1, 2, \ldots, b-a$. Also join every vertex in Y_j to every vertex in $Z_j, j = 1, 2, \ldots, a$, every vertex in Y_1 to every vertex in Y_2, every vertex in Y_2 to every vertex in Y_3, and w_j to every vertex in $Y_j, j = 1, 2, \ldots, a$. Finally, add the edges $w_j w_{jk}$ and $w_{j+k} w_{jk}$ for each $j = 1, 2, \ldots, b-a$ and for each $k = 1, 2, \ldots, a-j$.

To prove that $ir(G) = a$, we observe that A is irredundant. Now, let $x \in V - A$ and let $A_1 = A \cup \{x\}$. If $x \in Z_j$ or Y_j for some j, then w_j has no private neighbor with respect to A_1. If $x \in W_j$, then x itself has no private neighbor with respect to A_1. Thus A is a maximal irredundant set and hence $ir(G) \leq a$. Further any maximal irredundant set contains at least one vertex in $W_j \cup Y_j \cup Z_j, 1 \leq j \leq a$ and hence $ir(G) = a$.

We now proceed to prove that $\gamma(G) = b$. Clearly, $D = \{w_j : j = 1, 2, \ldots, b-a\} \cup Y$ is a dominating set of G and so $\gamma(G) \leq b$. Furthermore any dominating set D of G, it can be easily verified that $|D \cap (Y \cup Z)| \geq a$ and $|D \cap (A \cup W)| \geq b - a$. Hence $\gamma(G) = b$.

We now prove that $\gamma_{cc}(G) = c$. Clearly, $D' = \{w_{12}\} \cup \{w_{j1} : j = 2, \ldots, b-a\} \cup \{y_{11}, y_{21}\} \cup \{y_{j1} : j = 4, 5, \ldots, a\} \cup Z_3$ is a cc-dominating set of G and so $\gamma_{cc}(G) \leq c$. Furthermore any cc-dominating set D', it can be verified that $|D' \cap (Y \cup Z)| \geq a - 3 + 2 + c - b + 1 = c - b + a$ and $|D' \cap (A \cup W)| \geq b - a$. Hence $\gamma_{cc}(G) = c$.

We now prove that $i(G) = d$. Clearly, $I = \{w_{j1} : j = 1, 2, \ldots, b-a\} \cup \{y_{11}, y_{31}\} \cup \{y_{j1} : j = 4, 5, \ldots, a\} \cup Z_2$ is a maximal independent set of G and so $i(G) \leq d$. Further any independent set I of G, it can be verified that I contains at least $b - a$ vertices in $A \cup W$ and $|I \cap (Y \cup Z)| \geq d - b + a$. Thus $i(G) = d$.

Case ii. $b = a$.

We construct a graph G as in case(i) by taking $V(G) = A \cup Y \cup Z$ and replacing m by $d - b$ and n by $c - b$ in the above construction.Proceeding as in case(i), it can be proved that $ir(G) = a, \gamma(G) = b, \gamma_{cc}(G) = c$ and $i(G) = d$. ∎

4 Complexity Results

We prove that the decision problems corresponding to the parameters γ_{cc} and Γ_{cc} are NP-complete. One of the most common NP-complete problems is the exact cover by 3-sets problem ($X3C$), which was first shown to be NP-complete by Karp [5].

EXACT COVER BY 3-SETS ($X3C$)

INSTANCE. A finite set X with $|X| = 3q$ and a set $\mathcal{C}$ of 3-element subsets of X.

QUESTION. Does $\mathcal{C}$ contain an exact cover for X, that is a subcollection $C' \subseteq \mathcal{C}$ such that $|C'| = q$ and $\bigcup_{S \in C'} S = X$?

The decision problem for cc-dominating set can be stated as follows.

CC-**DOMINATING SET**

INSTANCE. A graph G and a positive integer k.

QUESTION. Does G have a cc-dominating set S with $|S| \leq k$?

Theorem 16 *cc-dominating set is NP-complete even when restricted to bipartite graphs.*

Proof. Obviously cc-dominating set is in NP. Now let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $\mathcal{C} = \{X_1, X_2, \ldots, X_r\}$ be an arbitrary instance of $X3C$. We construct a graph G as follows.

We take r copies G_i of $K_{1,3}$. Let v_i be the central vertex of G_i and label one of the leaves by X_i. Add vertices x_i for each element $x_i \in X$ and join the vertices x_i and x_j if $x_i \in X_j$. Let $k = r + q$.

Now, suppose $\mathcal{C}$ has an exact cover $C' = \{X_{i1}, X_{i2}, \ldots, X_{iq}\}$. Then $S = C' \cup \{v_1, v_2, \ldots, v_r\}$ is a dominating set, $|S| = r + q$, each v_i dominates 2 or 3 vertices in $V - S$, X_{ij} dominates 3 vertices in $V - S$ and each vertex in $V - S$ is dominated by exactly one vertex in S. Thus S is a cc-dominating set of G.

Now, suppose G contains a cc-dominating set S with $|S| \leq r+q$. Since $\gamma(G) \geq r+q$, it follows that $|S| = r + q$ and $v_i \in S, 1 \leq i \leq r$ and $x_i \notin S$, $1 \leq i \leq 3q$. Since each $X_i \in S$ dominates exactly three x_i's, it follows that $|S \cap \mathcal{C}| = q$ and $S \cap \mathcal{C}$ forms an exact cover for X. Hence cc-dominating set is NP-complete. ∎

Corollary 17 *Given a graph* G, *the problems of deciding whether* $\gamma(G) = \gamma_{cc}(G) = i(G)$ *or* $\gamma(G) = \gamma_{cc}(G)$ *or* $\gamma_{cc}(G) = i(G)$ *are NP-complete.*

Proof. For the graph G constructed in the Theorem 16, $\mathcal{C}$ has an exact cover if and only if $\gamma(G) = \gamma_{cc}(G) = i(G) = r + q$ and hence the result follows. ∎

3-SAT

INSTANCE. A set $X = \{x_1, x_2, \ldots, x_n\}$ of variables and a set $C = \{C_1, C_2, \ldots, C_j\}$ of 3 element sets called clauses, where each clause C_i contains three distinct occurrences of either a variable x_i or its complement x_i'.

QUESTION. Does C have a satisfying truth assignment?

UPPER CC-**DOMINATING SET**

INSTANCE. A graph G and a positive integer k.

QUESTION. Does G have a minimal cc-dominating set of size $\geq k$?

Theorem 18 *UPPER CC-DOMINATING SET is NP-complete.*

Proof. Clearly *cc*-dominating set is in NP. Given an instance C of 3-SAT, $X = \{x_1, x_2, \ldots, x_n\}$ and $C = \{C_1, C_2, \ldots, C_j\}$, we construct a graph G as follows.

We take n copies G_i of $K_{3,3}$, one for each literals x_i and j copies of H_i, one for each C_i. Label two adjacent vertices of G_i as x_i and x_i' and label one vertex of degree 3 in H_i as C_i (as shown in Figure 6 and Figure 7).Join the vertex labelled C_i with the vertices labelled x_k, x_l and x_m where $C_i = \{x_k, x_l, x_m\}$.

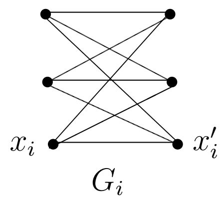

Figure 6

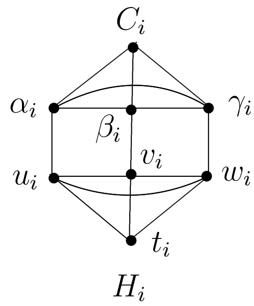

Figure 7

Suppose C has a satisfying truth assignment. Let D be the set consisting of all literals x_i or x_i' which are assigned the value true and the two vertices of G_i which are not dominated by the chosen vertex in D and the vertices u_i, v_i, w_i from H_i. Clearly, each component of $\langle D \rangle$ is K_1 or K_3 and $D - \{v\}$ is not a dominating set for all $v \in D$. Therefore D is a minimal *cc*-dominating set of order $3n + 3j$.

Conversely, suppose G contains a minimal *cc*-dominating set D with cardinality $\geq 3n + 3j$. Since D is minimal, it follows that $|D \cap V(G_i)| \leq 3$ and $|D \cap V(H_i)| \leq 3$. Since $|D| \geq 3n + 3j$, it follows that $|D \cap V(G_i)| = |D \cap V(H_i)| = 3$. Therefore, for $1 \leq i \leq j$, u_i, v_i and w_i are in D, exactly one of the vertices x_i, x_i' is in D and C_i does not belong to D. Hence C_i is dominated by exactly one of the vertices x_k, x_k'. Now define

$$f(x_i) = \begin{cases} T & \text{if } x_i \in D \\ F & \text{if } x_i \notin D. \end{cases}$$

Clearly, f is a satisfying truth assignment for the instance C. ∎

The following are some interesting problems for further investigation.

Problem 19 *For the hypercube Q_n, is $\gamma(Q_n) = \gamma_{cc}(Q_n)$ for $n \geq 6$?*

Problem 20 *Is $\gamma(P_n \square P_m) = \gamma_{cc}(P_n \square P_m)$?*

Problem 21 *Does there exist a linear time algorithm to compute $\gamma_{cc}(G)$ for trees?*

Problem 22 *Characterize graphs G for which $\gamma(G) = \gamma_{cc}(G)$.*

Problem 23 *Characterize graphs G for which $\Gamma(G) = \Gamma_{cc}(G)$.*

Problem 24 *Characterize graphs G for which $\Gamma_{cc}(G) = IR_{cc}(G) = \beta_{cc}(G)$.*

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b-coloring of Kneser graphs

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The b-chromatic number was introduced by R.W. Irving and D.F. Manlove [2] by considering proper colorings that are minimal with respect to a partial order defined on the set of all partitions of $V(G)$. They have shown that the determination of $b(G)$ is NP-complete for general graphs, but polynomial for trees. The b-chromatic number has received wide attention from the time of its introduction (See references at the end).

A k-coloring of a graph G is a b-coloring of G using k colors if each color class contains a color dominating vertex, that is, a vertex adjacent to at least one vertex of every other color class. The *b-chromatic number* of a graph G, denoted by $b(G)$, is the maximum k such that G has a b-coloring using k colors. Here also, as in the case of achromatic number, the chromatic number is the minimum k such that G has a b-coloring using k-colors.

The name b-chromatic number is analogous to the achromatic number. While the achromatic number of a graph G gives the maximum number of color classes in a 'complete' coloring of G, the b-chromatic number of a graph G gives the maximum number of color classes in a b-coloring of G.

It is known that for any $k \in [\chi(G), \psi(G)]$, there exists a complete coloring using k colors. But the same does not happen for b-coloring. That is, for some $k \in [\chi(G), b(G)]$, the graph G may not have a b-coloring using k colors. A graph G is said to be b-continuous if there exists a b-coloring of G using k colors for every $k \in [\chi(G), b(G)]$. It is an interesting question to find out as to which families of graphs are b-continuous. It has been proved that chordal graphs, sparse graphs, cographs and a particular type of planar graphs are all b-continuous.

The concept of Kneser graphs was introduced by M. Kneser in [4]. The Kneser graphs have drawn the attention of many graph theorists. It was conjectured by Kneser [4] in 1955 and proved by Lovász [6] in 1978 that $\chi(K(m, n)) = m - 2n + 2$. The proof was considered as an extraordinary achievement because it marked the beginning of the era of topological combinatorics: Applications of topological methods and theorems to problems of discrete mathematics.

The idea of finding the b-chromatic number of Kneser graphs was initiated in [3]. The b-chromatic number of Kneser graphs has been studied till date only for a few classes of graphs [3] and we list them below.

Theorem 1
For $n \geq 3$, $b(K(2n+1,n)) = n+2$.

Theorem 2
$K(2n+k,2)$ are b-continuous for $2n+k \geq 17$.

Theorem 3
For $m = 2n+k$, and $m \neq 8$,

$$b(K(m,2)) = \begin{cases} \left\lfloor \frac{m(m-1)}{6} \right\rfloor & \text{if } n \text{ is odd} \\ \left\lfloor \frac{(m-1)(m-2)}{6} \right\rfloor + 3 & \text{if } n \text{ is even} \end{cases}$$

and $b(K(8,2)) = 9$.

The natural question is: Are there Kneser graphs with $b = 1 + \Delta = 1 + d$, other than odd graphs? In the process of seeking an answer to this question, we have

Theorem 4
If G is a Kneser graph with $|V(G)| \leq 2d+2-2i$, then $b(G) \leq d-i$, where d is the degree of regularity of the Kneser graph.

For all the above graphs, $b < 1 + \Delta$.

Theorem 5
The odd graphs are b-continuous.

Now by Theorem 5, all odd graphs are b-continuous. Further, by Theorem 2, $K(2n+k,2)$ are b-continuous for $2n+k \geq 17$. Hence it is natural to ask the folowing question: Are the Kneser graphs b-continuous? For the Petersen graph, $\chi = 3 = b$, and hence it is trivially b-continuous. The following result shows that the question of examining b-continuity for other Kneser graphs is nontrivial.

Theorem 6
For any Kneser graph $G = K(2n+k,n)$ other than the Petersen graph, $b(G) \geq k+4 = \chi(G)+2$.

Theorem 7
If $3 \leq n \leq k+1$, then $b(K(2n+k,n)) \geq n+k+1$.

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Joint structural importance in consecutive-k-out-of-n systems

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The *joint reliability importance* $JRI(i,j)$ of two components i and j measures how these two components in a system interact in contributing to the system reliability $R(P)$. The joint reliability importance $JRI(i,j)$ is defined as follows.

$$JRI(i,j) = \frac{\partial^2 R(P)}{\partial p_i \partial p_j} = R(1_i, 1_j, P_{i,j}) + R(0_i, 0_j, P_{i,j}) - R(1_i, 0_j, P_{i,j}) - R(0_i, 1_j, P_{i,j}),$$

where p_i denotes the reliability of component i, P_i denotes all the component reliability except that of the component i, and 1_i and 0_i denote component i working and failed, respectively. The value of $JRI(i,j)$ is positive (negative) if and only if one component becomes more important (less important) when the other works. Joint reliability importance was first proposed independently by Hagstrom [2] and by Hong and Lie [5]. Note that the system reliability $R(P)$ can be computed only when the reliabilities of all components are well defined. Without the information of component reliabilities, we need to know the relative importance of the locations so that more reliable components can be assigned to the more important locations to maximize the system reliability. Many researchers have studied joint structural importance (setting all $p_i = p$) in many systems: the fault tree [3], the two-terminal system [1, 5, 7], the k-out-of-n system [5], etc.

A *consecutive-k-out-of-n system* consists of an ordered sequence of n components where the system fails if and only if any k consecutive components are all failed. Relative to low reliability of a series system and high reliability but very expensive hardware of the parallel system, the consecutive-k system has attracted many researchers. In this talk, we will present our results on the joint structural importance in consecutive-k-out-of-n systems.

We first state several natural symmetry properties of the joint structural importance in consecutive-k-out-of-n systems.

Lemma 1.

(i) $JSI(i,j) = JSI(j,i)$.

(ii) *$JSI(i,i) = 0$.*

(iii) *$JSI(i,j) = 0$ for $k > n$.*

(iv) *$JSI(i,j) = JSI(n-i+1, n-j+1)$.*

By Lemma 1, it suffices to discuss $JSI(i,j)$ in consecutive-k-out-of-n systems for $k \le n$ and $1 \le i \le \lceil n/2 \rceil$. Next, we consider $JSI(i,j)$ for two special cases: the series system ($k = 1$) and the parallel system ($k = n$).

Theorem 2. *Consider a consecutive-k-out-of-n system.*

(i) *For $k = 1$ (the series system), $JSI(i,j) = p^{n-2} > 0$.*

(ii) *For $k = n$ (the parallel system), $JSI(i,j) = -q^{n-2} < 0$.*

Next, we solve joint structural importance for $n \le 2k$.

Theorem 3. *Suppose $n \le 2k$.*

(i) *For $1 \le i \le n-k$,*

$$JSI(i,j) = \begin{cases} -jpq^{k-2} & \text{if } 1 \le j < i, \\ -ipq^{k-2} & \text{if } i < j \le k, \\ q^{k-1} + (j-k-i)pq^{k-2} & \text{if } k < j \le i+k, \\ 0 & \text{if } i+k < j \le n. \end{cases}$$

(ii) *For $n-k < i \le \lceil n/2 \rceil$,*

$$JSI(i,j) = \begin{cases} -jpq^{k-2} & \text{if } 1 \le j \le n-k, \\ -[(n-k)pq^{k-2} + q^{k-2}] & \text{if } n-k < j \le k, j \ne i, \\ -(n+1-j)pq^{k-2} & \text{if } k < j \le n. \end{cases}$$

In the following, we discuss joint structural importance for $n > 2k$. First, we consider $i = 1$.

Theorem 4.

(i) *For $2 \le j \le k$, $JSI(1,j) < 0$. Furthermore, $JSI(1,j) = -pq^{k-2}R(n-k-1)$.*

(ii) *For $j = k+1$, $JSI(1,j) > 0$. Furthermore, $JSI(1,k+1) = q^{k-1}R(n-k-1)$ for $n > 2k$ and $JSI(1,k+1) = q^{k-1}$ for $k+1 \le n \le 2k$.*

(iii) *For $j \ge k+2$, $JSI(1,j) = pq^{k-2}(R(j-k-2)R(n-j) - R(n-k-1))$. Furthermore, $JSI(1,j) > 0$ for $n > 2k$ and $JSI(1,j) = 0$ for $k+2 \le j \le n \le 2k$.*

Theorem 5. *$JSI(1, j) = JSI(1, n + k + 2 - j)$ for $k + 2 \leq j \leq n$.*

Theorem 6. *For $2 \leq j \leq min\{i+k+1, n\}$, the ordering of $JSI(1, j)$ are $JSI(1, j') = JSI(1, k) < 0 < JSI(1, n) = JSI(1, k+2) < JSI(1, j) < JSI(1, k+1)$, for $2 \leq j' \leq k - 1$ and $k + 3 \leq j \leq n - 1$.*

Next, consider $i > 1$.

Theorem 7. *For $2 \leq i < j \leq k + 1$, $JSI(i - 1, j) - JSI(i, j) = pq^{k-2}R(n - i - k)$. Furthermore, $JSI(i-1, j) > JSI(i, j)$ for $n \geq i+k-1$ and $JSI(i-1, j) = JSI(i, j)$ for $n \leq i + k - 2$.*

Theorem 8. *Consider $k + 2 \leq j < 2k$.*

(i) *For $k+1 \leq i < j < 2k$, $JSI(i-1, j) - JSI(i, j) = pq^{k-2}R(i-2)R(n-i-k) + pq^{k-2}R(i - k - 2)(R(n - j) - R(n - i)) > 0$.*

(ii) *For $j - k + 1 \leq i \leq k$ and $k + 2 \leq j < 2k$, $JSI(i - 1, j) - JSI(i, j) = pq^{k-2}R(n - i - k) > 0$.*

Theorem 9. *For $j - k + 1 \leq i < j$ and $2k \leq j \leq \lceil n/2 \rceil$, $JSI(i - 1, j) - JSI(i, j) = pq^{k-2}R(i - 2)R(n - i - k) + pq^{k-2}R(i - k - 2)(R(n - j) - R(n - i)) > 0$.*

Theorem 10. *Consider $i < k$. $JSI(i, j) < JSI(i, j + 1)$ for $k \leq j \leq i + k - 1$. Furthermore, $JSI(i, j+1) - JSI(i, j) = pq^{k-2}R(j - k - 1)R(n - j - 1) + pq^{k-2}R(n - j - k - 1)(R(i - 1) - R(j - 1)) > 0$.*

Theorem 11. *$JSI(i, i + k) > JSI(i, i + k + 1)$ and $JSI(i, i - k) > JSI(i, i - k - 1)$ for $n \geq i + 2k$.*

Note that, given a fixed i, the graph of $JSI(i, j)$ has a W-shape property for $max\{1, i - k - 1\} \leq j \leq min\{n, i + k + 1\}$ with $JSI(i, i) = 0$.

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On a new class of graphs from posets: The cover-incomparability graphs

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In this paper we introduce a new graph that can be associated to a poset P, we call it the cover-incomparability graph (C-I graph) of P. This is the graph in which the edge set is the union of the edge sets of the corresponding cover graph and the corresponding incomparability graph. Note that this is the only nontrivial way to construct a new associated graph as unions and/or intersections of the edge sets of the three standard associated graphs. Our motivation for C-I graphs comes from the theory of transit functions that can in particular be studied on posets.

The notion of transit functions was introduced by Mulder about ten years ago and finally written up in [4]. The central idea of this concept is to generalize the interval function of a graph [3], and to study how to move around in discrete structures. A *transit function* on a non empty set V is a function $T : V \times V \to 2^V$ satisfying the following transit axioms:

(t1) $u \in T(u, v)$ for any u and $v \in V$.

(t2) $T(u, v) = T(v, u)$ for all u and $v \in V$.

(t3) $T(u, u) = \{u\}$ for all $u \in V$.

The *underlying graph* G_T of a transit function T on a set V is the graph with vertex set V, where distinct u and v in V are joined by an edge if $\mid T(u, v) \mid= 2$.

For a poset $P = (V, \leq)$, the *standard poset transit function* $T_P : V \times V \to 2^V$ is defined in the following way:

(i) If x and y are incomparable, then $T_P(x, y) = \{x, y\}$.

(ii) If $x \leq y$, then $T_P(x, y) = \{z \mid x \leq z \leq y\}$.

(iii) If $y \leq x$, then $T_P(x, y) = \{z \mid y \leq z \leq x\}$.

Clearly, T_P satisfies (t1)-(t3). In other words, T_P is a transit function.

The underlying graph G_{T_P} of T is obtained from the cover graph of P by adding an edge between any pair of incomparable elements of P. Thus the edges of G_{T_P} are the union of the edges of the cover graph of P and the incomparability graph of P. Hence we say that G_{T_P} is the *cover-incomparability graph* (*C-I graph*) of P.

Let $P = (V, \leq)$ be a poset. If $u \leq v$ but $u \neq v$, then we write $u < v$. If u and v are in V, then v *covers* u in P if $u < v$ and there is no w in V with $u < w < v$. If $u \leq v$ we will sometimes say that u is *below* v, and that v is *above* u. Let V' be a nonempty subset of V. Then there is a natural poset $Q = (V', \leq')$, where $u \leq' v$ if and only if $u \leq v$ for any $u, v \in V'$. The poset Q is called a *subposet* of P and its notation is simplified to $Q = (V', \leq)$. If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an *isometric subposet.* For the purposes of this paper we will say that P is called *Q-free* if P has no isometric subposet isomorphic to Q.

The main results that we discuss are actually special instances of the general theorem given below. Some of these results already appeared in the paper [1], which is the first paper on these classes of graphs. In another paper following this paper, it is proved that the recognition complexity of cover-incomparability graphs is NP-complete, [2].

Theorem Let $\mathcal{G}$ be a class of graphs with a forbidden induced subgraphs characterization. Let

$$\mathcal{P} = \{P \mid P \text{ is a poset with } G_{T_P} \in \mathcal{G}\}.$$

Then $\mathcal{P}$ has a forbidden isometric subposets characterization.

As special instances of this theorem, we characterize C-I graphs which belong to some of the standard graph families which possess forbidden induced subgraphs characterization. We specifically discuss the C-I graphs which are chordal, distance heriditary, Ptolemiac. Also we address the reverse problem, namely among some standard classes of graphs, which graphs are cover-incomparability graphs. This we address for block graphs and split graphs. This problem is important for the standard graph families whose recognition complexity is polynomial.

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Total and connected domination in circulant graphs

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Let Γ be a finite group with e as the identity. A generating set of the group Γ is a subset A such that every element of Γ can be expressed as the product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. The Cayley graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, y)_a | x, y \in V(G)$, there exists $a \in A$ such that $y = xa\}$ and it is denoted by $Cay(\Gamma, A)$. The exclusion of e from A eliminates the possibility of loops in the graph. The inclusion of the inverse in A for every element of A means that an edge is in the graph regardless of which end vertex is considered. For $x, y \in V(G)$, there exists $g \in \Gamma$ such that $y = xg$. One can express g as product of $a_1, a_2, ..., a_n \in A$. Then y and x are connected by a path through $a_1, a_2, ..., a_n \in A$. Hence G is connected and $|A|$ is the degree of $Cay(\Gamma, A)$.

Suppose G is a graph, the open neighbourhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of vertices adjacent to v. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood $N(S)$ is defined to be $\cup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$ [3]. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or adjacent to an element of S [3]. A dominating set S is a minimal dominating set if no proper subset is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of all dominating sets in G [3] and the corresponding dominating set is called a γ-set. A dominating set S is an *independent dominating set* if no two vertices in S are adjacent. The *independent domination number* $i(G)$ of a graph G is the minimum cardinality of all independent dominating sets in G [1]. The *domatic number* $d(G)$ of a graph G to be the maximum number of elements in a partition of $V(G)$ into dominating sets. The *independent domatic number* $d_i(G)$ of a graph G is the maximum number of elements in a partition of $V(G)$ into independent dominating sets [3]. Similarly the *perfect domatic number* $d_p(G)$ is the maximum number of elements in a partition of $V(G)$ into perfect dominating sets of G. A graph G is said to be excellent if each vertex u of G is contained in some γ-set of G. The graph G is said to be k-excellent, if every subset S of $V(G)$ with $|S| = k$ is contained in some γ-set of G. Tamizh Chelvam and Rani [5, 6] have obtained the domination number and independent domination number for certain circulant graphs. A set $S \subset V$ is a total dominating set if $N(S) = V$, or equivalently, if for every vertex $v \in V$, there exists a vertex $u \in S,\ u \neq v$, such that u is adjacent to v. The total domination number $\gamma_t(G)$ is the minimum cardinality among all total dominating sets of G [3].

A total dominating set of a graph G with cardinality $\gamma_t(G)$ is called a γ_t-set of G. A graph G is a total excellent graph if to each $u \in V$, there is a γ_t-set of G containing u. A dominating set S is a connected dominating set if $< S >$ is a connected subgraph of G. The connected domination number $\gamma_c(G)$ is the minimum cardinality among all the connected dominating sets of G [3] and the corresponding set is denoted by γ_c-set of G.

Throughout this paper, n is a fixed positive integer, $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and $G = Cay(\mathbb{Z}_n, A)$, where A is a generating set of $\mathbb{Z}_n$. Hereafter $+$ stands for modulo n addition in $\mathbb{Z}_n$. Unless otherwise specified A stands for the set $\{1,\ n-1,\ 2,\ n-2, \ldots,\ k,\ n-k\}$ where $1 \le k \le \frac{n-1}{2}$.

Theorem 1. [5] *Let $G = Cay(\mathbb{Z}_n, A)$ where $A = \{1, n-1, 2, n-2, \ldots, k, n-k\}$ and n, k are positive integers with $1 \le k \le \frac{n-1}{2}$. Then $\gamma(G) = \lceil \frac{n}{|A|+1} \rceil$. Further $D = \{0, (2k+1), 2(2k+1), 3(2k+1), \ldots, (\ell-1)(2k+1)\}$, where $\ell = \lceil \frac{n}{|A|+1} \rceil$ is a γ-set of G.*

Theorem 2. [6] *Let n and k be positive integers such that $k \le \frac{n-1}{2}$ and $2k+1$ divides n. Then $i(G) = \gamma_p(G) = \frac{n}{2k+1}$, where $G = Cay(\mathbb{Z}_n, A)$.*

Theorem 3. [6] *If n, k are positive integers such that $k \le \frac{n-1}{2}$ and $(2k+1)$ does not divide n, then $i(G) = \lceil \frac{n}{2k+1} \rceil$.*

Theorem 4. [6] *Let n and k be positive integers such that $k \le \frac{n-1}{2}$. Then $ii(G) = \gamma i(G) = \gamma\gamma(G) = 2\lceil \frac{n}{2k+1} \rceil$, where $G = Cay(\mathbb{Z}_n, A)$.*

In this paper, we obtain the value of total and connected domination numbers for certain circulant graphs and the results are given below.

Theorem 5. *Suppose $n (\ge 3)$ and k are positive integers with $1 \le k \le \frac{n-1}{2}$. Assume that $n = (\ell_t - 1)(3k+1) + h$ where $\ell_t = \lceil \frac{n}{3k+1} \rceil$ and for some h with $1 \le h \le k$. Let $A = \{1, 2, \ldots, k, n-k, \ldots, n-1\}$ and $G = Cay(\mathbb{Z}_n, A)$. Then $\gamma_t(G) = 2\lceil \frac{n}{3k+1} \rceil - 1$.*

Theorem 6. *Let $n (\ge 3)$ and k be positive integers with $1 \le k \le \frac{n-1}{2}$ and $3k+1$ divides n. Let $A = \{1, 2, \ldots, k, n-k, \ldots, n-1\}$ and $G = Cay(\mathbb{Z}_n, A)$. Then $\gamma_t(G) = 2\lceil \frac{n}{3k+1} \rceil$.*

Theorem 7. *Let $n (\ge 3)$ be a positive even integer and k is an integer such that $1 \le k \le \frac{n-2}{2}$. Let $A = \{1, 2, \ldots k, n-k, \ldots, n-1, \frac{n}{2}\}$and $G = Cay(\mathbb{Z}_n, A)$. Then $\gamma_t(G) = 2\lceil \frac{n}{4k+2} \rceil$.*

Theorem 8. *Suppose n and k are positive integers with $1 \le k \le \frac{n-1}{2}$. Let $A = \{1, 2, \ldots, k, n-k, \ldots, n-1\}$ and $G = Cay(\mathbb{Z}_n, A)$. Then $\gamma_c(G) = \lceil \frac{n-(2k+1)}{k} \rceil + 1$.*

Theorem 9. *Let n be an even positive integer and k is a positive integer such that $1 \le k \le \frac{n-1}{2}$. Let $A = \{1, 2, \ldots, k, n-k, \ldots, n-1, \frac{n}{2}\}$ and $G = Cay(\mathbb{Z}_n, A)$. Then $\gamma_c(G) = \lceil \frac{n-2(2k+1)}{k} \rceil + 2$.*

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Constructing independent spanning trees for hypercubes and locally twisted cubes

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The use of multiple independent spanning trees (ISTs) for data broadcasting in networks provides a number of advantages such as the increase of fault-tolerance and bandwidth. Thus the designs of multiple ISTs in several classes of networks have been widely investigated. In [24], Zehavi and Itai stated two versions of the n independent spanning trees conjecture. The vertex (edge) conjecture is that any n-connected (n-edge-connected) graph has n vertex-ISTs (edge-ISTs) rooted at an arbitrary vertex r. In [14], Khuller and Schieber proved that the vertex conjecture implies the edge conjecture. Recently, in [10], Hsieh and Tu proposed an algorithm to construct n edge-ISTs rooted at vertex 0 for an n-dimensional locally twisted cube LTQ_n, which is a variant of the hypercube. Since LTQ_n is it not vertex-transitive, Hsieh and Tu's result does not solve the edge conjecture for the locally twisted cube. In the paper, we confirm the vertex conjecture (and hence also the edge conjecture) for the locally twisted cube by proposing an algorithm to construct n vertex-ISTs rooted at any vertex for the LTQ_n. We also confirm the vertex conjecture (and hence also the edge conjecture) for the hypercube.

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Cyclic sieving for cyclic polytopes

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In [8], Reiner-Stanton-White introduced the following enumerative phenomenon for a set of combinatorial structures under an action of a cyclic group.

Let X be a finite set, $X(q)$ a polynomial in $\mathbf{Z}[q]$ with the property $X(1) = |X|$, and C a finite cyclic group acting on X. The triple $(X, X(q), C)$ is said to exhibit the *cyclic sieving phenomenon* (CSP) if for every $c \in C$,

$$[X(q)]_{q=\omega} = |\{x \in X : c(x) = x\}|, \tag{1}$$

where ω is a root of unity of the same multiplicative order as c. Such a polynomial $X(q)$ implicitly carries the information about the orbit-structure of X under C-action. Namely, if $X(q)$ is expanded as $X(q) \equiv a_0 + a_1 q + \cdots + a_{n-1} q^{n-1} \pmod{q^n - 1}$, where n is the order of C, then a_k counts the number of orbits whose stabilizer-order divides k.

Consider the moment curve $\gamma : \mathbf{R} \to \mathbf{R}^d$ defined parametrically by $\gamma(t) = (t, t^2, \ldots, t^d)$. For any n real numbers $t_1 < t_2 < \cdots < t_n$, let

$$P = \mathrm{conv}\{\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n)\}$$

be the convex hull of the n distinct points $\gamma(t_i)$ on γ. Such a polytope is called a *cyclic polytope* of dimension d. It is known that the points $\gamma(t_i)$ are the vertices of P and the combinatorial equivalence class (with isomorphic face lattices) of polytopes with P does not depend on the specific choice of the parameters t_i (see [9]).

Let $\mathsf{CP}(n, d)$ denote a d-dimensional cyclic polytope with n vertices. Among the d-dimensional polytopes with n vertices, the cyclic polytope $\mathsf{CP}(n, d)$ is the one with the greatest number of k-faces for all $0 \le k \le d-1$ (by McMullen's upper bound, see [9, Theorem 8.23]). Let $f_k(\mathsf{CP}(n, d))$ be the number of k-faces of $\mathsf{CP}(n, d)$. These numbers were first determined by Motzkin [5] but no proofs were given. For a proof using the Dehn-Sommerville equations, see [3, Section 9.6]. A combinatorial proof was given by Shephard [7].

Theorem 1. ([7, Corollary 2]) *For $1 \le k \le d$, the number $f_{k-1}(\mathsf{CP}(n, d))$ of $(k-1)$-faces of $\mathsf{CP}(n, d)$ is given by $f_{k-1}(\mathsf{CP}(n, d)) = \sum_{j=1}^{\frac{d}{2}} \frac{n}{n-j} \binom{n-j}{j} \binom{j}{k-j}$ if d is even, or*

$\sum_{j=1}^{\frac{d+1}{2}} \frac{k+1}{j}\binom{n-j}{j-1}\binom{j}{k+1-j}$ *if* d *is odd, with the usual convention that* $\binom{n}{m} = 0$ *if* $n < m$ *or* $m < 0$.

Moreover, Kaibel and Waßmer [4] derived the automorphism group of $\mathsf{CP}(n, d)$.

Theorem 2. ([4]) *The combinatorial automorphism group of* $\mathsf{CP}(n, d)$ *is isomorphic to one of the following groups:*

	$n = d + 1$	$n = d + 2$	$n \geq d + 3$
d even	S_n	$S_{\frac{n}{2}} \text{ wr } \mathbf{Z}_2$	D_n
d odd	S_n	$S_{\lceil \frac{n}{2} \rceil} \times S_{\lfloor \frac{n}{2} \rfloor}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$

where S_n *is the symmetric group of order* n *and* D_n *is the dihedral group of order* n.

Consider the cyclic group $C = \mathbf{Z}_n$, generated by $c = (1, 2 \ldots, n)$, acting on $\mathsf{CP}(n, d)$ by cyclic translation of the vertices, according to the order on the curve γ. By Gale's evenness condition, it turns out that the cyclic group C is an automorphism subgroup of $\mathsf{CP}(n, d)$ if and only if either $n = d + 1$ or d is even. One of the main results in this paper is to prove the CSP for faces of $\mathsf{CP}(n, d)$ for even d, under C-action. For even d and $1 \leq k \leq d$, we define

$$F(n, d, k; q) = \sum_{j=1}^{\frac{d}{2}} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k-j \end{bmatrix}_q, \tag{2}$$

with the usual convention that $\begin{bmatrix} n \\ m \end{bmatrix}_q = 0$ if $n < m$ or $m < 0$. Clearly, $F(n, d, k; 1) = f_{k-1}(\mathsf{CP}(n, d))$.

Theorem 3. *For even* d *and* $1 \leq k \leq d$, *let* X *be the set of* $(k-1)$*-faces of* $\mathsf{CP}(n, d)$, *let* $X(q) = F(n, d, k; q)$ *be the polynomial defined in Eq. (2), and let* $C = \mathbf{Z}_n$ *act on* X *by cyclic translation of the vertices. Then the triple* $(X, X(q), C)$ *exhibits the cyclic sieving phenomenon.*

For odd d, C is no longer an automorphism group of $\mathsf{CP}(n, d)$ for $n \geq d + 2$. Inspired by the work of Kaibel and Waßmer [4], we consider automorphism groups C' and C'' of order 2 of $\mathsf{CP}(n, d)$, which are generated by $c' = (1, n)(2, n-1) \cdots$ and $c'' = (1, n)$, respectively. In an attempt on proving the CSP, we derive the number of $(k-1)$-faces that are C'-invariant (or C''-invariant). However, so far it lacks a feasible option for the q-polynomial. It is worth mentioning that the natural q-analogue does not work in this situation.

For the C''-case, one may come up with an artificial polynomial $X(q)$ serving as the polynomial for the CSP. However, the downside is that the polynomial $X(q)$ is always of degree 1 and has no good connection with the group C''.

However, In particular, for $n = d + 2$, from the automorphism group $S_{\lceil \frac{n}{2} \rceil} \times S_{\lfloor \frac{n}{2} \rfloor}$ of $\mathsf{CP}(n, d)$, we present two (rather curious) instances of CSP, under the group

$\mathbf{Z}_{\lceil \frac{n}{2} \rceil}$ (resp. $\mathbf{Z}_{\lfloor \frac{n}{2} \rfloor}$) that cyclically translates the odd-positioned (resp. even-positioned) vertices, along with feasible q-polynomials.

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On optimal pebbling of hypercubes

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Let G be a graph such that $V(G) = \{v_1, v_2, \ldots, v_p\}$. By a *distribution* of pebbles on G we mean a function $\delta : V(G) \to N \cup \{0\}$ and for clarity, we use $(\delta_{v_1}, \delta_{v_2}, \ldots, \delta_{v_p})$ to denote δ, where δ_v is the number of pebbles distributed on $v \in V(G)$. The *support* S_δ of δ is defined as the set of vertices v in $V(G)$ such that $\delta_v > 0$. Therefore the number of pebbles used in G is $\sum_{v \in S_\delta} \delta_v$ and denoted by δ_G.

A *pebbling move* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a distribution δ of pebbles lets us move at least one pebble to each vertex v by applying pebbling moves repeatedly(if necessary), then δ is called a pebbling of G. The *optimal pebbling number* $f'(G)$ of G is the minimum number of pebbles used in a pebbling of G. Note here that the *pebbling number* $f(G)$ of G is the minimum number of pebbles k such that any distribution of k pebbles is a pebbling of G. See [1, 5] for references.

The problem of pebbling graph was first proposed by J. Lagarias and M. Saks as a tool for solving a number theoretic problem by Lemke and Kleitman [5]. Since then, quite a few of work has been done by F. R. K. Chung [1], Guzman, Moews [9], Pachter [9], Clarke et. al. [2] and Herscovici et. al. [4]. Its dual concept, the optimal pebbling number of a graph G was first introduced by Pachter [9] and the following results are notable.

Theorem 1 **[9]** *Let P be a path with $3t + r$ vertices with $0 \le r \le 2$. Then $f'(P) = 2t + r$.*

In [10], Fu and Shiue extend the study and they find the optimal pebbling number for the caterpillar.

Theorem 2 **[11]** *For any two graphs G and H, $f'(G \times H) \le f'(G)f'(H)$.*

Theorem 3 [3] *Let T_h^m be a complete m-ary tree with height h. Then* $f'(T_h^m) = 2^h$ *for each* $m \geq 3$, *and* $f'(T_h^2) = \min\{\sum_{i=0}^{h} 2^i x_i | \sum_{i=0}^{h}(2^i - \frac{1}{3})x_i \geq \frac{1}{3} \cdot 2^{h+1}, x_0 \in \{0,1,2,3\}$ *and* $x_i \in \{0,2\}$, *where* $i = 1, 2, \ldots, h\}$.

Theorem 4 [8] *Let Q_n be the hypercube defined by* $Q_n = Q_{n-1} \times K_2$. *Then* $f'(Q_n) = (\frac{4}{3})^{n+O(\log n)}$.

In fact, the upper bound of $f'(Q_n)$ obtained by Moews [8] is as follows.

Corollary 5 [8] $f'(Q_n) \leq 2(\frac{4}{3})^n n^2$.

In this talk, we present a newly obtained result on hypercubes which improves this upper bound.

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On joint triangulations of two sets of points in the plane

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Abstract

In this paper, we establish two necessary conditions for a joint triangulation of two sets of n points in the plane and conjecture that they are sufficient. We show that these necessary conditions can be tested in $O(n^3)$ time. For the problem of a joint triangulation of two simple polygons of n vertices, we propose an $O(n^3)$ time algorithm for constructing a joint triangulation using dynamic programming.

1 Introduction

Let S be a set of points in the plane. A triangulation of S is a maximal set of line segments with endpoints in S such that no two segments intersect in their interior. A triangulation of S partitions the convex hull of S into regions not containing points in S that are bounded by triangles. Triangulating a set of points in the plane under various constraints is a well studied problem in computational geometry [1, 5]. Here we consider the problem of triangulating two sets of points jointly.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two disjoint sets of points in the plane, specified by their respective x and y coordinates. A line segment b_ib_j is called the *corresponding line segment* of the line segment a_ia_j and vice versa. Similarly, a triangle $b_ib_jb_k$ is called the *corresponding triangle* of $a_ia_ja_k$ and vice versa. Let $\mathcal{T}(A)$ and $\mathcal{T}(B)$ denote the set of all triangulations of A and B. The problem of joint triangulation of A and B is to find triangulations $T(A) \in \mathcal{T}(A)$ and $T(B) \in \mathcal{T}(B)$, if they exist, such that for each region bounded by a triangle $a_ia_ja_k$ in $T(A)$, the corresponding triangle $b_ib_jb_k$ bounds a region in $T(B)$ (see Figure 1). The problem

Figure 1: The joint triangulation of two sets of points $A = \{a_1, a_2, \ldots, a_8\}$ and $B = \{b_1, b_2, \ldots, b_8\}$.

was posed in 1987 by Saalfeld [6] and it has an application in automated cartography. Over the last two decades, several researchers have worked on this problem but the problem is still open.

The above definition of a joint triangulation of A and B needs some clarification. Consider triangulations $T(A)$ and $T(B)$ of point sets A and B respectively, shown in Figure 2. It can be seen that for every line segment a_ia_j in $T(A)$, the corresponding line segment b_ib_j is in $T(B)$ and vice versa. However, the triangle $a_4a_5a_6$ does not contain any point of A, whereas the corresponding triangle $b_4b_5b_6$ contains points of B. Thus the triangles bounding the regions are different and we do not consider this to be a joint triangulation. This gives rise to the definition of a component triangle as defined by Saalfeld [6]. A triangle in $T(A)$ or $T(B)$ is said to be a *component triangle* of the triangulation if it does not contain any point in its interior. Therefore, the problem of joint triangulation of A and B is to compute $T(A)$ and $T(B)$, if they exist, such that a triangle $a_ia_ja_k$ is a component triangle in $T(A)$ if and only if the corresponding triangle $b_ib_jb_k$ is a component triangle in $T(B)$.

In the next section, we propose two necessary conditions for this problem and conjecture that they are sufficient. We also present an $O(n^3)$ time algorithm for testing these necessary conditions. If the given set of points A and B satisfy the two necessary conditions, we propose different algorithms for constructing joint triangulations of A and B in Section 3. The proposed algorithms have been implemented and experimental results suggest that the algorithms correctly construct joint triangulations of A and B whenever A and B satisfy the two necessary conditions. In Section 4, we present an $O(n^3)$ time algorithm for computing a joint triangulation of two simple polygons of n vertices. In Section 5, we conclude the paper with a few remarks.

Figure 2: The triangle $a_4a_5a_6$ does not contain any point of A, whereas the corresponding triangle $b_4b_5b_6$ contains points of B.

2 Necessary conditions

Let $CH(A)$ and $CH(B)$ denote the boundary of convex hulls of A and B respectively. We state the first necessary condition for the existence of a joint triangulation of A and B, which relates the edges of $CH(A)$ and $CH(B)$,

Necessary condition 1: If there exists a joint triangulation of A and B, then a_ia_j is an edge of $CH(A)$ if and only if the corresponding edge b_ib_j is an edge of $CH(B)$.

Proof: Assume on the contrary that there is a joint triangulation of A and B and an edge a_ia_j is an edge in $CH(A)$ but the corresponding edge b_ib_j is not an edge in $CH(B)$. Since a_ia_j is an edge of $CH(A)$, there exists only one component triangle (say, $a_ia_ja_k$) with a_ia_j as an edge, in any triangulation of A. On the other hand, we know that any joint triangulation must include b_ib_j in the triangulation of B. Since b_ib_j is not an edge in $CH(B)$ by assumption, there are two component triangles (say, $b_ib_jb_k$ and $b_ib_jb_l$) with b_ib_j as an edge, in the triangulation of B. Since the component triangle $a_ia_ja_l$ is not present in the triangulation of A, this contradicts the definition of a joint triangulation.

A triangle $a_ia_ja_k$ is said to be an *empty triangle* in A if it does not contain any point of A in its interior. Let S_A denote the set of all empty triangles in A whose corresponding triangles in B are empty triangles in B. Let S_B be the set of triangles corresponding to the triangles in S_A. It follows from the definition of a joint triangulation that only triangles from S_A and S_B can be component triangles in a joint triangulation of A and B. Let $a_ia_ja_k$ and $a_ia_ja_l$ be two triangles in S_A such that they lie on opposite sides of their common edge a_ia_j. If $b_ib_jb_k$ and $b_ib_jb_l$ also lie on opposite sides of their common edge b_ib_j, then $a_ia_ja_l$ is a called a *successor triangle*

Figure 3: On the edge a_6a_7, $a_6a_7a_8$ and $a_6a_7a_2$ are successor triangles. The corresponding triangles $b_6b_7b_8$ and $b_6b_7b_2$ are also successor triangles on the edge b_6b_7

of $a_ia_ja_k$ on the edge a_ia_j and vice versa. Analogously, $b_ib_jb_l$ is also called a *successor triangle* of $b_ib_jb_k$ on the edge b_ib_j and vice versa. In Figure 3, $a_6a_7a_8$ and $a_6a_7a_2$ are successor triangles on the edge a_6a_7 and their corresponding triangles $b_6b_7b_8$ and $b_6b_7b_2$ are also successor triangles on the edge b_6b_7. On the other hand, $a_6a_7a_8$ and $a_2a_7a_8$ are not successor triangles on the edge a_7a_8 as a_2 and a_6 lie on the same side of a_7a_8. Since successors of $a_ia_ja_k$ and $b_ib_jb_k$ are defined jointly, in what follows, we say that ijl is a successor triangle of ijk on edge ij and vice versa. Observe that ijk can have more than one successor triangle on an edge ij. In Figure 3, $(2, 6, 8)$, $(7, 6, 8)$ and $(3, 6, 8)$ are successor triangles of $(5, 6, 8)$ on the edge $(6, 8)$. It is obvious that there is no successor triangle on any edge of the convex hull.

Intuitively, if a triangle ijk is a component triangle in a joint triangulation, one of the successors on each edge of ijk that is not a convex hull edge is also a component triangle in the joint triangulation. Let S denote the maximal subset of triangles in S_A and S_B such that each triangle ijk in S has at least one successor triangle in S, on the edges ij, jk and ki that are not convex hull edges. Note that if a triangle ijk does not have a successor triangle on a non convex hull edge, then ijk can not belong to S. We call triangles in S as *legal triangles* and S is called the set of legal triangles. Now, we state the second necessary condition.

Necessary condition 2: If there exits a joint triangulation of A and B, then the set of legal triangles S is not empty.

Proof: If there is a joint triangulation of A and B, then every component triangle in the joint triangulation has a successor triangle on each its non convex hull edges. So, every component triangle in a joint triangulation is a legal triangle and hence, the set of legal triangles S is not empty.

Conjecture: There exists a joint triangulation of A and B if and only if A and B satisfy the two necessary conditions.

Let us present an algorithm for testing the necessary conditions. The first necessary condition can be tested by traversing the boundary of the convex hulls of A and B. Since the convex hulls can be computed in $O(n \log n)$ time [1, 5], the first necessary condition can be tested in $O(n \log n)$ time.

For testing the second necessary condition, the algorithm starts by computing all empty triangles in A and B. It has been shown by Dobkin et. al [13] that all empty triangles in a set of n points in a plane can be computed in time proportional to the number of empty triangles which can be at most $O(n^3)$. So, S_A and S_B can be computed in $O(n^3)$ time.

For every non-convex hull edge ij of all triangles in S_A and S_B, the algorithm checks whether there exists two triangles ijk and ijl on the edge ij in S_A as well as in S_B such that k and l lie on opposite sides of ij in both A and B. If ij satisfies this condition, then there are successor triangles on the edge ij. Otherwise, all triangles in S_A and S_B with ij as an edge are removed from S_A and S_B, and the remaining two edges of every deleted triangle are pushed into a queue Q. For each edge ef in Q, check whether there are successor triangles on ef. If the condition is satisfied, then ef is removed from the queue. Otherwise, all triangles in S_A and S_B with ef as an edge are removed from S_A and S_B, and the remaining edges of every deleted triangles are pushed into the queue Q. This process is repeated till either S_A and S_B become empty or the queue becomes empty. In the latter case, all remaining triangles in S_A and S_B have successors on all non-convex hull edges, in which case they form the set of legal triangles S. Note that that the cost of processing edges in Q can be assigned to deleted triangles which can be at most $O(n^3)$. We state the result in the following theorem.

Theorem 1: Given two sets A and B of n points in the plane, the two necessary conditions for a joint triangulation of A and B can be tested in $O(n^3)$ time.

3 Algorithms for constructing joint triangulations

In this section, we present two algorithms for finding a joint triangulation of A and B which run in $O(n^3)$ time. We assume that the set of legal triangles S has been computed by the algorithm as mentioned in the previous section. If the set S is empty, clearly no joint triangulation exists. So, we consider the other case when S is not empty.

Constructing a joint triangulation of A and B involves finding a subset T of legal

Figure 4: A joint triangulation of two simple polygons A and B.

triangles in S forming a triangulation in A and the corresponding triangulation in B. The algorithm uses a greedy method to obtain T. Initialize $S' = S$ and $T = \emptyset$. Take any triangle ijk from S', add it to T and delete all triangles in S' that intersect the interior of the triangle ijk in either A or B. Repeat this process until S' becomes empty. Our claim is that the triangles in T form a joint triangulation of A and B. We have been unable to prove this claim, which would also prove the sufficiency of the two necessary conditions. On the other hand, we have observed experimentally that whenever S is not empty, the algorithm always finds a joint triangulation of A and B.

An alternative algorithm for finding a joint triangulation is as follows. An edge ij of a triangle is called a *legal edge* if it is an edge of some legal triangle ijk in S. Let L denote the set of all legal edges and T be any maximal subset of L such that no two edges in T intersect in their interior. We have observed that T gives a joint triangulation of A and B. Again, we have been unable to prove this claim.

The set of legal triangles and legal edges seem to satisfy some nice properties. One observation is that two legal triangles ijk and pqr intersect in their interior in A if and only if they intersect in their interior in B. The same property seems to hold for legal edges as well. Thus, to find a joint triangulation, it seems sufficient to find a triangulation of A consisting of only legal edges, or one in which the component triangles are legal triangles. However, we have been unable to prove any of these properties.

4 Computing a joint triangulation of two simple polygons

In this section, we present an $O(n^3)$ time algorithm for computing a joint trian-

Figure 5: Testing the sub-polygon $Q_{1,4}$ for a joint triangulation.

gulation of two simple polygons $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ using dynamic programming. Two points u and v in a simple polygon are said to be *visible* if the line segment uv lies totally inside the polygon. Let $VG(A)$ denote the visibility graph of A, where vertices of A are vertices of $VG(A)$ and two vertices in $VG(A)$ are connected by an edge if and only if the corresponding vertices in A are visible in A [3]. The visibility graph $VG(B)$ of B is defined analogously. We have the following observation (see Figure 4).

Lemma 1: All edges of the triangles in a joint triangulation of A and B must belong to $VG(A)$ and $VG(B)$ respectively.

Let $IVG(A)$ denote the sub-graph of $VG(A)$ such that an edge $a_i a_j$ of $VG(A)$ belongs to $IVG(A)$ if and only if $b_i b_j$ is an edge of $VG(B)$. Analogously, we define $IVG(B)$. It follows from Lemma 1 that we have to consider only the edges of $IVG(A)$ and $IVG(B)$ in a joint triangulation of A and B. Since the visibility graph of a simple polygon can be computed in time proportional to the number of edges in the visibility graph, which can be at most $O(n^2)$ [30], $IVG(A)$ and $IVG(B)$ can be computed in $O(n^2)$ time.

Let $SUB(A)$ denote the set of all sub-polygons of A (including A itself) that can be formed by cutting A using only one diagonal of $IVG(A)$. So, the size of sub-polygons in $SUB(A)$ varies from 3 to n. We use a boolean function $M(Q)$ to indicate whether a sub-polygon Q admits joint triangulation. Since all sub-polygons of three vertices in $SUB(A)$ (say, $Q_{1,3}, Q_{2,3}, \ldots$) admit joint triangulations as they are triangles, $M(Q_{1,3}), M(Q_{2,3}), \ldots$ are set to be true. Then the procedure considers sub-polygons $Q_{1,4}, Q_{2,4}, \ldots$ of $SUB(A)$ having four vertices. Let $Q_{1,4} = (a_i, a_{i+1}, a_{i+2}, a_{i+3})$ (see Figure 5). So, $a_i a_{i+3}$ is the diagonal of $IVG(A)$ used to cut A to form $Q_{1,4}$. Let a_k be a vertex of $Q_{1,4}$ such that edges $a_i a_k$ and $a_k a_{i+3}$ belong to $IVG(A)$. If no such v_k exists, then set $M(Q_{1,4})$ to false. If $a_{i+1} = a_k$ and the triangle $(a_{i+1}, a_{i+2}, a_{i+3})$ admits triangulation found in the previous step, then set $M(Q_{1,4})$ to true. If $a_{i+2} = a_k$ and the triangle (a_i, a_{i+1}, a_{i+2}) admits triangulation found in the previous step, then set

$M(Q_{1,4})$ to true. Otherwise, set $M(Q_{1,4})$ to false.

Similarly, the procedure considers sub-polygons $Q_{1,5}, Q_{2,5}, \ldots$ of $SUB(A)$ having five vertices by locating all possible such vertices a_k. This process is repeated till the sub-polygon of size n (i.e., A) is considered. In the following, we state the major steps of the procedure.

Step 1: Divide A into sub-polygons using diagonals of $IVG(A)$ to form $SUB(A)$;

Step 2: Consider each edge of A as a degenerated triangle; *For* each edge $a_i a_{i+1}$ *do* $M(a_i a_{i+1}) := true$;

Step 3: *For* each sub-polygon $Q_{j,3}$ of size three in $SUB(A)$ *do* $M(Q_{j,3}) := true$; $size := 4$;

Step 4: *For* each sub-polygon $Q_{j,size}$ in $SUB(A)$ *do*

Step 4.1: *If* $Q_{j,size} = A$ *then* $i := 1$, $q := n$, $k := 2$ and *goto* Step 4.3;

Step 4.2: Let $a_i a_q$ be the diagonal used to cut A to form $Q_{j,size} = (a_i, a_{i+1}, \ldots, a_q)$; $k := i + 1$;

Step 4.3: *If* $a_i a_k$ and $a_q a_k$ are edges in $IVG(A)$ and two sub-polygons formed by removing the triangle (a_i, a_k, a_q) from $Q_{j,size}$ admit joint triangulations *then* $M(Q_{j,size}) := true$;

Step 4.4: *If* $k \neq q - 1$ *then* $k := k + 1$ and *goto* Step 4.3;

Step 5: *If* $size \neq n$ then $size := size + 1$ and *goto* Step 4;

Step 6: *If* $M(A)$ is true *then* by backtracking identify diagonals of $IVG(A)$ giving a joint triangulation *else* report that there is no joint triangulation.

Step 7: Stop.

Since the procedure uses triangles formed by edges of $IVG(A)$ and $IVG(B)$, and these triangles are added one at a time (i.e., $a_i a_k a_q$) to verify whether a joint triangulation exists for the sub-polygons formed by the union of triangles verified so far, the procedure correctly computes a joint triangulation of A and B if it exists. Since the number of sub-polygons in $SUB(A)$ can be at most $O(n^2)$ and the procedure can take $O(n)$ time for testing each sub-polygon, the overall time required by the algorithm is $O(n^3)$. We state the result in the following theorem.

Theorem 2: Given two simple polygons A and B of n points, a joint triangulation of A and B can be constructed in $O(n^3)$ time.

5 Concluding remarks

We conclude in this section by mentioning some extensions of the basic problem.

An immediate extension is to consider finding a joint triangulation of k sets of labeled points. It is easy to verify that for such a joint triangulation to exist, boundary of the convex hulls of all sets of points must contain the same edges. Further, the notion of a successor triangle can be extended to any number of sets of points in a natural way. A triangle ijl is a successor of a triangle ijk on the edge ij if and only if it is a successor in all point sets. Thus we may define the set of legal triangles in an analogous way. We believe that the same conjecture holds for any number of sets of points.

Further generalizations are possible by considering triangulations of objects other than just point sets. In particular, we can consider triangulations of any connected polygonal region with points and polygonal holes inside. The only difference here is that a triangle containing an edge of a hole boundary may not have a successor on that edge. Thus one necessary condition is that the hole boundaries must contain the same set of edges in all point sets. The definition of a successor triangle and a legal triangle may be modified accordingly, and the same algorithms can also be used. Again, we have observed empirically that if the set of legal triangles is not empty, there exists a joint triangulation, and it may be constructed in the same greedy fashion as for two point sets.

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Unsolved problems in visibility graph theory

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1 Introduction

The visibility graph is a fundamental structure studied in the field of computational geometry, and pose some special challenges [12, 26]. Apart from theoretical interests, visibility graphs has important applications also. Some of the early applications include computing Euclidean shortest paths in the presence of obstacles [36] and decomposing two-dimensional shapes into clusters [40]. For more on the uses of this class of graphs, see [37, 43]. Let P be a simple polygon with or without holes in the plane. We say two points a and b in P are *mutually visible* if the line segment ab lies entirely within P. This definition allows the segment ab to pass through a reflex vertex or graze along a polygonal edge. The *visibility graph* (also called the *vertex visibility graph*) G of P is defined by associating a node with each vertex of P such that (v_i, v_j) is an undirected edge of G if polygonal vertices v_i and v_j are mutually visible. Figure 1(b) shows the visibility graph of the polygon in Figure 1(a). Sometimes the visibility graph is drawn directly on the polygon, as shown in Figure 1(c). It can be seen that every triangulation of P corresponds to a subgraph of the visibility graph of P.

Figure 1: (a) A polygon. (b) The visibility graph of the polygon.
(c) The visibility graph drawn on the polygon.

The problem of computing the visibility graph of a polygon P (with or without holes) having a total of n vertices is well studied [32, 36, 41]. Asano et al. [6] and Welzl [49] proposed $O(n^2)$ time algorithms for this problem. Since, at its largest, a visibility graph can be of size $O(n^2)$, algorithms of Asano et al. and Welzl are worst-case optimal. The visibility graph may be much smaller than its worst-case size of $O(n^2)$ (in particular, it can have $O(n)$ edges) and therefore, it is not necessary to spend $O(n^2)$ time to compute it. Hershberger [30] developed an $O(E)$ time output sensitive algorithm for computing the visibility graph of a simple polygon. Ghosh and Mount [28] presented $O(n \log n + E)$ time, $O(E + n)$ space algorithm for computing visibility graph of polygon with holes. Keeping the same time complexity, Pocchiola and Vegter [39] improved the space complexity to $O(n)$.

2 Visibility Graph Recognition, Characterization, and Reconstruction

Consider the opposite problem of determining if there is a polygon P whose visibility graph is the given graph G. This problem is called the visibility graph *recognition* problem. Identifying the set of properties satisfied by all visibility graphs is called the visibility graph *characterization* problem. The problem of actually drawing one such polygon P whose visibility graph is the given graph G, is called the visibility graph *reconstruction* problem.

2.1 Visibility Graph Recognition

The general problem of recognizing a given graph G as the visibility graph of a simple polygon P is yet to be solved. However, this problems has been solved for visibility graphs of *spiral* polygons [18, 19] and *tower* polygons [9].

Open Problem 1 *Given a graph G in adjacency matrix form, determine whether G is the visibility graph of a simple polygon P.*

Open Problem 2 *Is the problem of recognizing visibility graphs in NP?*

Ghosh [24] has presented four necessary conditions for recognizing visibility graphs of simple polygons under the assumption that a Hamiltonian cycle of the given graph, which corresponds to the boundary of the simple polygon, is given as input along with the graph. It has been pointed out by Everett and Corneil [18, 20] that these conditions are not sufficient as there are graphs that satisfy the three necessary conditions but they are not visibility graphs of any simple polygon. These counterexamples can be eliminated once the third necessary condition is strengthened. It have been shown by Srinivasraghavan and Mukhopadhyay [45] that the stronger version of the third necessary condition proposed by Everett [18] is in fact necessary. On the other hand, the counterexample given by Abello, Lin and Pisupati [4] shows that the three necessary conditions of Ghosh [24] are not sufficient even with the stronger version of the third necessary condition. In a later paper by Ghosh [25], another necessary condition is identified which circumvents the counterexample of Abello, Lin and Pisupati [4]. Ghosh's necessary conditions are as follows.

Necessary condition 1. *Every ordered cycle of $k \geq 4$ vertices in a visibility graph has at least $k-3$ chords.*

Necessary condition 2. *Every invisible pair in a visibility graph has at least one blocking vertex.*

Necessary condition 3. *There is an assignment in a visibility graph such that no blocking vertex v_a is assigned to two or more minimal invisible pairs that are separable with respect to v_a.*

Necessary condition 4. *Let D be any ordered cycle of a visibility graph. For any assignment of blocking vertices to all minimal invisible pairs in the visibility graph, the total number of vertices of D assigned to the minimal invisible pairs between the vertices of D is at most $|D|-3$.*

It has been shown later by Streinu [46, 47] that these four necessary conditions are also not sufficient. It is not clear whether another necessary condition is required to circumvent the counter example.

Open Problem 3 *Given a graph G in adjacency matrix form along with a Hamiltonian cycle of G, determine whether G is the visibility graph of a simple polygon P whose boundary corresponds to the given Hamiltonian cycle.*

2.2 Visibility Graph Characterization

Let us state some results on the problems of characterizing visibility graphs for special classes of simple polygons. The earliest result is from ElGindy [17] who showed that every *maximal outerplanar graph* is a visibility graph of a simple polygon, and he suggested an $O(n \log n)$ algorithm for reconstruction. If all reflex vertices of a simple polygon occur consecutively along its boundary, the polygon is called a *spiral polygon*. Everett and Corneil [18, 19] characterized visibility graphs of spiral polygons by showing that these graphs are a subset of *interval graphs* which lead to an $O(n)$ time algorithm. Choi, Shin and Chwa [9] characterized funnel-shaped polygons, also called *towers*, and gave an $O(n)$ time recognition algorithm. Visibility graphs of towers are also characterized by Colley, Lubiw and Spinrad [10] and they have shown that visibility graphs of towers are bipartite permutation graphs with an added Hamiltonian cycle. If the internal angle at each vertex of a simple polygon is either 90 or 270 degrees, then the polygon is called a *rectilinear polygon*. If the boundary of a rectilinear polygon is formed by a staircase path with two other edges, the polygon is called a *staircase polygon*. Visibility graphs of staircase polygons have been characterized by Abello, Eğecioğlu, and Kumar [1]. Lin and Chen [34] have studied visibility graphs that are *planar*.

For the characterization of visibility graphs of arbitrary simple polygons, Ghosh has shown that visibility graphs do not possess the characteristics of *perfect graphs*, *circle graphs* or *chordal graphs*. On the other hand, Coullard and Lubiw [11] have proved that every triconnected component of a visibility graph satisfies *3-clique ordering*. This property suggests that structural properties of visibility graphs may be related to well-studied graph classes, such as *3-trees* and *3-connected graphs*. Everett and Corneil [18, 20] have shown that there is no finite set of forbidden induced subgraphs that characterize visibility graphs. Abello and Kumar [2, 3] have suggested a set of necessary conditions for recognizing visibility graphs. However, it has been shown in [25] that this set of conditions follow from the last two necessary conditions of Ghosh.

Open Problem 4 *Characterize visibility graphs of simple polygons.*

2.3 Visibility Graph Reconstruction

Let us mention some of the approaches on the visibility graph reconstruction problem. It has been shown by Everett [18] that visibility graph reconstruction is in *PSPACE*. This is the only upper bound known on the complexity of the problem. Abello and Kumar [3] studied the relationship between visibility graphs and oriented matroids, Lin and Skiena [35] studied the equivalent order types, and Streinu [46, 47] and O'Rourke and Streinu [38] studied psuedo-line arrangements. Everett and Corneil [18, 20] have solved the reconstruction problem for the visibility graphs of *spiral* polygons and the corresponding problem for the visibility graph of *tower* polygons has been solved by Choi, Shin and Chwa [9]. Reconstruction problem with added information has been studied by Coullard and Lubiw [11], Everett, Hurtado, and Noy [21], Everett, Lubiw, and O'Rourke [22], Jackson and Wismath [31].

Open Problem 5 *Given the visibility graph G of a simple polygon, draw a simple polygon whose visibility graph is G.*

3 Graph Theoretic Problems on Visibility Graphs

3.1 Hamiltonian Cycle in Visibility Graphs

A *Hamiltonian cycle* is a cycle in an undirected graph which visits each vertex exactly once and also returns to the starting vertex. The Hamiltonian cycle problem is to determine whether a Hamiltonian cycle exists in a given graph G. Observe that G may contain several Hamiltonian cycles, and G may be visibility graph for a Hamiltonian cycle and is not a valid visibility graph for another Hamiltonian cycle in G.

Open Problem 6 *Given the visibility graph G of a simple polygon P, determine the Hamiltonian cycle in G that corresponds to the boundary of P.*

3.2 Minimum Dominating Set in Visibility Graphs

A *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is joined to at least one member of D by some edge. The minimum dominating set problem in visibility graphs corresponds to the art gallery problem in polygons which has been shown to be NP-hard [33, 35]. Following the approximation algorithm for the art gallery problem for polygons given by Ghosh [23], a minimum dominating set of visibility graph can be computed with an approximation ratio of $O(\log n)$.

Open Problem 7 *Design a constant factor approximation algorithm for computing minimum dominating set of visibility graphs.*

3.3 Maximum Hidden Set in Visibility Graphs

An *independent set* is a set of vertices in a graph no two of which are adjacent. Independent sets in visibility graphs are known as *hidden vertex sets*. Shermer [42] has proved that the maximum hidden vertex set problem on visibility graphs is also NP-hard. However, the problem may not remain NP-hard if the Hamiltonian cycle corresponding to the boundary of the simple polygon is given as an input along with the visibility graph. With this additional input, Ghosh, Shermer, Bhattacharya and Goswami [29] have shown that it is possible to compute in $O(ne)$ time the maximum hidden vertex set in the visibility graph of a very special class of simple polygons called *convex fans*, where n and e are the number of vertices and edges of the input visibility graph of the convex fan respectively. Hidden vertex sets are also studied by Eidenbenz [14, 15], Ghosh et al. [27] and Lin and Skiena [35].

Open Problem 8 *Given the visibility graph G of a simple polygon P along with the Hamiltonian cycle in G corresponding to the boundary of P, determine the maximum hidden set of G.*

Open Problem 9 *Design an approximation algorithm for computing maximum hidden set of visibility graph.*

3.4 Maximum Clique in Visibility Graphs

A *clique* in a graph is a set of pairwise adjacent vertices. The problem of computing the maximum clique in the visibility graph is not known to be NP-hard. Observe that the maximum clique in a visibility graph corresponds to the largest empty convex polygon inside the corresponding polygon. Algorithms for computing largest empty convex polygons has been reported by several authors [8, 13, 16]. However, for each of these algorithms, the input is either a polygon [16] or a point set [8, 13]. Spinrad [44] has discussed possible approaches for computing maximum clique using the notion of *triangle-extendible ordering* which is essentially a transitive orientation of the graph.

Open Problem 10 *Given a visibility graph G in adjacency matrix form, compute a maximum clique of G.*

Open Problem 11 *Determine whether a set of vertices of a visibility graph has a triangle-extendible ordering in polynomial time.*

If the Hamiltonian cycle in a visibility graph corresponding to the boundary of the polygon is given along with the visibility graph as an input, Ghosh, Shermer, Bhattacharya and Goswami [29] have presented an $O(n^2e)$ time algorithm for computing the maximum clique in the visibility graph G of a simple polygon P. Here n and e are number of vertices and edges of G respectively.

4 Counting Visibility Graphs

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if and only if there is a bijection f that maps vertices of V_1 to the vertices of V_2 such that an edge $(v, w) \in E_1$ if and only if the edge $(f(v), f(w)) \in E_2$. It has been shown [26] that the number of *non-isomorphic* visibility graphs of simple polygons of n vertices is at least 2^{n-4}. On the other hand, a straight forward application of Warren's theorem [48] shows that the number of visibility graphs is at most $2^{O(n \log n)}$ [44].

Let G_1 and G_2 be the visibility graphs of simple polygons P_1 and P_2 respectively. Let C_1 (or C_2) denote the Hamiltonian cycles in G_1 (respectively, G_2) that corresponds to the boundary of P_1 (respectively, P_2). Polygons P_1 and P_2 are called *similar* if and only if there is a bijection f that maps adjacent vertices on the boundary of P_1 to that of boundary of P_2 such that $f(G_1) = G_2$ [35]. It has been shown [7] that similarity of P_1 and P_2, each of n vertices, can be determined in $O(n^2)$ time. Therefore, given G_1

and G_2 along with C_1 and C_2, the corresponding visibility graph similarity problem can also be solved in $O(n^2)$ time. It has been shown by Lin and Slkiena [35] that two simple polygons with isomorphic visibility graphs may not be similar polygons.

5 Representing Visibility Graphs

Although the most natural form of representation for visibility graphs would be to use coordinates of the points, this is not useful if we are looking for a space efficient representation. Lin and Skiena [35] have proved that visibility graphs require endpoints to have exponential sized integers. However, it is not known whether singly exponential sized integers are sufficient. It is important because if we could guarantee that the number of bits in the integer is polynomial, then visibility graph recognition is in NP [44].

Open Problem 12 *Can all endpoints of a visibility graph be assigned integer coordinates such that the integers use a polynomial number of bits?*

A natural form of storage is studied by Agarwal et al. [5] which uses a relatively small number of bits to store a visibility graph. However, the representation is neither space optimal, nor adjacency information can be retrieved in constant time. However, it is the most significant reduced space representation which is currently known. The authors consider the problem of representing a visibility graph as a covering set of cliques and complete bipartite graphs so that every graph in the set is a subset of G, and every edge is contained in at least one of the graphs of the covering set. Their proposed algorithm constructs a covering set which has $O(n \log^4 n)$ bits. It can be shown that any covering set requires $\Omega(n \log^2 n)$ bits on some visibility graphs [44]. Given this gap between upper and lower bounds on this natural form of representation, we have a number of problems.

Open Problem 13 *Give a tight bound (with respect to order notation) on the number of bits used in an optimal covering set of a visibility graph.*

Open Problem 14 *Find a covering set which matches the above bound in polynomial time.*

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On the equitable colorings of Kneser graphs

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An m-coloring of a graph G is a mapping $f : V(G) \to \{1, 2, \ldots, m\}$ such that $f(x) \neq f(y)$ for any two adjacent vertices x and y in G. A graph G is m-colorable if it admits an m-coloring. The *chromatic number* $\chi(G)$ of G is the minimum number m such that G is m-colorable. The well-known Brooks' Theorem is stated as following.

Theorem 1[1]. Suppose G is a graph different from a complete graph and an odd cycle. Then $\chi(G) \leq \Delta(G)$.

An equitable m-coloring of a graph G is an m-coloring such that any two color classes differ in size by at most one. A graph G is equitably m-colorable if it admits an equitable m-coloring. The *equitable chromatic number* $\chi_=(G)$ of G is the minimum number m such that G is equitably m-colorable. One can also consider the minimum number m such that G is equitably r-colorable for all $r \geq m$. Such a number m is called the *equitable chromatic threshold* of G, denoted by $\chi^*_=(G)$. It is clear that $\chi(G) \leq \chi_=(G) \leq \chi^*_=(G)$. Since $\chi(G) \leq \chi_=(G)$, Meyer then posed the following conjecture which, if true, is stronger than the Brooks' Theorem.

Conjecture 1 [11]. Suppose G is a connected graph different from a complete graph and an odd cycle. Then $\chi_=(G) \leq \Delta(G)$.

One well-known result of Hajnal and Szemerédi, when rephrased in terms of the equitable colorability, has already been shown as follows.

Theorem 2 [4, 5]. A graph G, not necessary connected, is equitably m-colorable if $m \geq \Delta(G) + 1$.

Theorem 2 says that $\chi_=(G) \leq \chi^*_=(G) \leq \Delta(G) + 1$ for all graphs G. Since the graphs G that require at least $\Delta(G) + 1$ colors to color the vertices equitably are complete graphs and odd cycles, Chen, Lih and Wu put forth the following.

Conjecture 2 [3]. (Equitable Δ-Coloring Conjecture) A connected graph G is equitably $\Delta(G)$-colorable if and only if G is different from the complete graph K_n, the odd cycle C_{2n+1} and the complete bipartite graph $K_{2n+1,2n+1}$ for all $n \geq 1$.

They also verified this conjecture for a graph with $\Delta(G) \geq \frac{|V(G)|}{2}$ or $\Delta(G) \leq 3$. Yap and Zhang[1] obtained a finer bound when $\frac{|V(G)|}{2} > \Delta(G) \geq \frac{|V(G)|}{3} + 1$. Moreover, some particular cases have been studied, such as trees[**?**, 2], bipartite graphs[9], d-degenerate graphs[7, 8] and planar graphs[6, 12, 13]. However, Conjecture 1 and Conjecture 2 are still open in general.

For $n \geq 2k+1$, the *Kneser graph* $KG(n,k)$ has the vertex set consisting of all k-subsets of an n-set. Two distinct vertices are adjacent in $KG(n,k)$ if they have empty intersection as subsets. The *Odd graph* O_k is the Kneser graph $KG(2k+1,k)$. The chromatic number of $KG(n,k)$ was obtained by Lovász.

Theorem 3 [10]. $\chi(KG(n,k)) = n - 2k + 2$.

Since $KG(n,1) = K_n$, it is easy to see that $\chi(KG(n,1)) = \chi_=(KG(n,1)) = \chi_=^*(KG(n,1)) = n$. In this talk, we study the equitable colorings of $KG(n,k)$ for $k \geq 2$. Mainly, we obtain the following results.

Theorem 4. Suppose that $m \geq n-k+1$. Then $KG(n,k)$ is equitably m-colorable, that is, $\chi_=(KG(n,k)) \leq \chi_=^*(KG(n,k)) \leq n-k+1$.

Theorem 5. If $\left\lfloor \frac{C(n,k)}{n-k} \right\rfloor > \alpha_2(n,k) = C(n-1,k-1) - C(n-k-1,k-1) + 1$, then $\chi_=(KG(n,k)) = \chi_=^*(KG(n,k)) = n-k+1$.

Theorem 6. For $n \geq 5$,

$$\chi_=(KG(n,2)) = \chi_=^*(KG(n,2)) = \begin{cases} n-1 & \text{if } n \geq 7, \\ n-2 & \text{if } n = 5 \text{ or } 6. \end{cases}$$

Theorem 7. For $n \geq 7$,

$$\chi_=(KG(n,3)) = \chi_=^*(KG(n,3)) = \begin{cases} n-2 & \text{if } n \geq 16, \\ n-3 & \text{if } 14 \leq n \leq 15, \\ n-4 & \text{if } 7 \leq n \leq 13. \end{cases}$$

Theorem 8. $\chi(O_k) = \chi_=(O_k) = \chi_=^*(O_k) = 3$ for $k \geq 1$.

We conclude this talk by posing the following conjecture.

Conjecture 3. Is it true that $\chi_=(KG(n,k)) = \chi_=^*(KG(n,k))$ for $k \geq 2$?

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An extension of Stein-Lovász Theorem and some of its applications

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Let X be a finite set and Γ be a family of subsets of X. We denote by $H = (X, \Gamma)$ the hypergraph having X as the set of vertices and Γ as the set of hyperedges. A subset $T \subseteq X$ such that $T \bigcap E \neq \emptyset$ for any hyperedge E is called a *cover* of the hypergarph H. The minimum size of a cover of an hypergraph H is denoted by $\tau(H)$. An upper bound for $\tau(H)$ was given by Lovász [9]:

$$\tau(H) \leq \frac{|X|}{\min_{E\in\Gamma} |E|}(1 + \ln \triangle),$$

where $\triangle = \max_{x\in X} |\{E : E \in \Gamma \text{ with } x \in E\}|$.

An equivalent statement in terms of the block-point incidence matrices of the corresponding hypergraphs was given by Stein [10] independently. It was called the Stein-Lovász Theorem in [6] while dealing with the covering problems in coding theory. The Stein-Lovász theorem was first used in dealing with the upper bounds for the sizes of (k, m, n)-selectors [2]. Inspired by this work, it was also used in dealing with the upper bound for the sizes of $(d, r; z]$-disjunct matrices [5]. Some more applications can also be found in [8]. The notion of (k, m, n)-selectors was first introduced by de Bonis et al. in [2], followed by a generalization to the notion of (k, m, c, n)-selectors [3].

Theorem [8] Let A be a (0,1) matrix with N rows and M columns. Assume that each row contains at least v ones, and each column at most a ones. Then there exists an $N \times K$ submatrix C with

$$K \leq (\frac{N}{a}) + (\frac{M}{v}) \ln a \leq (\frac{M}{v})(1 + \ln a),$$

such that C does not contain an all-zero row.

The greedy procedure as shown in the proof provides the desired submatrix one column at a time, and hence the algorithm below follows.

Algorithm: STEIN-LOVÁSZ(A)
input: A is an $N \times M$ matrix, each column has at most a ones, each row has at least v ones

$C \leftarrow \emptyset$

while A has at least one row

do find a column c in A having maximum weight

delete all rows of A that contain a "1" in column c

delete column c from A

output: Returns a submatrix of A with no all-zero row

The Stein-Lovász theorem can be further extended from rows with weight at least 1 to the case of rows with weight at least $z \geq 1$. The bound can be further improved when M is a matrix with constant row weight and column weight as well, i.e., in the language of hypergraphs, uniform and regular.

At each state, a new column is added to the submatrix that maximizes the number of "new" rows that are yet uncovered. When all rows are covered, the algorithm stops. It seems quite interesting that we can use the Stein-Lovász Theorem to derive bounds for some combinatorial array. If the entries of 1 occur in a binary matrix of order $N \times M$ as uniformly as possible, for example, v-uniform and a-regular, an upper bound for K is found so that the submatrices of order $N \times K$ will share similar property too.

Main Theorem Let A be a (0,1) matrix of order $N \times M$, and let a, v, z be positive integers. Assume that each row contains exactly v ones, and each column exactly a ones. Then there exists a submatrix C of order $N \times K$ with

$$K \leq \frac{v}{v-(z-1)}\, z(\frac{M}{v})(1+\ln a),$$

such that each row of C has weight at least z. More specifically, if the matrix is v-uniform and a-regular, the upper bound can be reduced to

$$K \leq z(\frac{M}{v})(1+\ln a).$$

The strategy for the proof is as follows:

1. Use the Stein-Lovász Theorem to obtain a submatrix C_1 with each row weight at least 1.

2. Choose some columns in the matrix $A \backslash C_1$ to combine with the submatrix C_1 to form a submatrix C_2 with each row weight at least 2.

3. Choose some columns in the matrix $A\backslash C_2$ to combine with the submatrix C_2 to form a submatrix C_3 with each row weight at least 3.

4. Step by step, and finally we obtain the desired submatrix $C = C_z$ with each row weight at least z.

Note that this upper bound makes sense only if $\frac{v}{v-(z-1)} z(\frac{M}{v})(1 + \ln a) < M$, i.e., $z < \frac{v+1}{2+\ln a}$ in general, or if $z(\frac{M}{v})(1 + \ln a) < M$, i.e., $z < \frac{v}{1+\ln a}$ for the case of uniform and regular.

For the case of uniform and regular, the main theorem is restated in the language of hypergraphs in the following corollary. A z-cover is a subset C of X such that $|C \bigcap E| \geq z$ for any hyperedges E. Let $\tau_z(H)$ be the minimum size over all z-covers of the hypergraph H.

Corollary Let $H = (X, \Gamma)$ be a v-uniform and a-regular hypergraph with vertex set X and edge set Γ, then $\tau_z(H) \leq z\frac{|X|}{v}(1 + \ln a)$.

We conjecture that $\tau_z(H) \leq z\tau_1(H)$ holds for hypergraphs which are uniform and regular. However, it need not be true in general as shown in the following example. For the hypergraph with where $X = \{1, 2, 3, ..., 8\}$ and $\Gamma = \{\{1, 2, 3\}, \{4, 5, 6\}, \{1, 7, 8\}\}$. It is easy to see that $\{1, 4\}$ is a 1-cover with minimum size, hence $\tau_1(H) = 2$. Similarly, $\{1, 2, 4, 5, 7\}$ is a 2-cover with minimum size, hence $\tau_2(H) = 5$. This shows that $\tau_2(H) = 5 > 2 \cdot 2 = 2\tau_1(H)$.

The extended Stein-Lovász theorem will be used in dealing the upper bound for the sizes of $(d, s$ out of $r; z]$-disjunct matrices and $(k, m, c, n; z)$-selectors. It in turn provides a few upper bounds for the sizes in various models with specific parameters. Those upper bounds for the sizes of uniform splitting systems, uniform separating systems, cover designs and lotto designs are given respectively.

Most of these bounds are roughly the same as those derived by the basic probabilistic method including the Lovász Local Lemma. Thus, due to its constructive nature, the Stein-Lovász theorem can be regarded as a de-randomized algorithm for the probabilistic methods. The relationship between the extended Stein-Lovász theorem and the Lovász Local Lemma deserve further study.

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On $\{P_2 \cup P_3, C_4\}$-free graphs

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In 1965, Fulkerson and Gross [4] showed that any chordal graph on n vertices has at most n maximal cliques. Since then several more classes of graphs have been identified which contain $p(n)$ number of maximal cliques, where $p(n)$ denotes a polynomial in n; see [1, 2, 3, 5, 6]. It follows that the "clique number decision problem" for these graphs can be solved in polynomial time, while for a general class of graphs this problem is known to be NP-complete.

In this paper, we derive a decomposition theorem for the class of $\{P_2 \cup P_3, C_4\}$-free graphs by using the well known concept of *expansion* of graphs.

Let G be a graph on n vertices $v_1, v_2, \ldots, v_n$, and let $H_1, H_2, \ldots, H_n$ be any n vertex disjoint graphs. Then an *expansion* $G(H_1, H_2, \ldots, H_n)$ of G is the graph obtained from G by

(i) replacing the vertex v_i of G by H_i, $i = 1, 2, \ldots, n$, and

(ii) joining the vertices $x \in H_i$, $y \in H_j$ iff v_i and v_j are adjacent in G.

An expansion is also called a *composition*; see [7].

Theorem 1. *If G is a connected $\{P_2 \cup P_3, C_4\}$-free graph, then G is either chordal or there exists a partition (V_1, V_2, V_3) of $V(G)$ such that (i) $[V_1] \cong K_m^c$, for some $m \geq 0$, (ii) $[V_2] \cong K_t$, for some $t \geq 0$, (iii) $[V_3]$ is isomorphic to a graph obtained from one of the graphs shown in Figure 1, by expanding each vertex indicated in circle by a complete graph (of order ≥ 1) and each vertex indicated in square by a complete graph (of order ≥ 0), (iv) $[V_1, V_3] = \emptyset$, and (v) $[V_2, V_3 \setminus S]$ is complete (see Figure 1 for the set S).*

Corollary 1. If G is a $\{P_2 \cup P_3, C_4\}$-free graph, then there exists a vertex v such that $[N[v]]$ is chordal.

Using Theorem 1, we show the following results.

Theorem 2. *There exists an $O(nm)$ algorithm to recognize a $\{P_2 \cup P_3, C_4\}$-free graph.*

Theorem 3. *If G is a $\{P_2 \cup P_3, C_4\}$-free graph, then* $\chi(G) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$.

Theorem 4. *If G is a $\{P_2 \cup P_3, C_4\}$-free graph, then G has at most $n + 5$ maximal cliques. The upper bound is attained if and only if G is the Petersen graph.*

Using Corollary 1, we have the following.

Theorem 4. *If G is a $\{P_2 \cup P_3, C_4\}$-free graph, then there exists an $O(n^2 m)$ algorithm to generate all the maximal cliques in G.*

If every vertex in a graph G belongs to at most k maximal cliques, then G has at most $\frac{kn}{2}$ maximal cliques. So, we deduce the following result for a class of graphs which is larger than the class of graphs of Theorem 4.

Theorem 5. *If G is a $\{K_1 + (P_2 \cup P_3), K_1 + C_4\}$-free, then G has at most $\frac{n^2 + 5n}{2}$ maximal cliques.*

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A primal-dual algorithm for the unconstrained fractional matching problem

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The problem of finding the maximum matching for a graph is a well-studied problem in graph theory for which efficient algorithms have been developed for both bipartite graphs as well as general graphs [1, 2]. In addition, the *maximum fractional matching problem*, a relaxed version of the general matching problem, allows edges to exist "fractionally" in the matching [3, 4, 5].

In this abstract, we present a combinatorial algorithm for a variant of the *Maximum Matching Problem* which allow negative edge values and arbitrary capacities on the vertices. We solve the proposed problem, called the *Unconstrained Fractional Matching Problem*, by characterizing the optimal solution to the dual of the restricted primal (DRP) and give a combinatorial algorithm for solving the DRP.

The primal-dual algorithm [7] relies on the relationship between a given linear program P and its dual D. Starting with a feasible solution π for the dual D, the primal-dual algorithm works towards feasibility in the primal. When the objective function values of feasible solutions for P and for D are equal, then we have an optimal solution to the primal-dual pair. The primal ($\mathrm{P_{ufm}}$) and dual ($\mathrm{D_{ufm}}$) linear programs for the unconstrained fractional matching problem are given below:

$$\begin{array}{lll} \textit{minimize} & \sum_{u\in V} c_u x_u & \\ \textit{subject to} & x_u + x_v = 1 & \forall (u,v)\in E \\ & x_u \geq 0 & \forall u \in V \end{array} \qquad (\mathrm{P_{ufm}})$$

$$\begin{array}{lll} \textit{maximize} & \sum_{e\in E} \pi_e & \\ \textit{subject to} & \sum_{e\in E(u)} \pi_e \leq c_u & \forall u\in V \\ & \pi_e \gtrless 0 & \forall e \in E \end{array} \qquad (\mathrm{D_{ufm}})$$

For a feasible solution π for ($\mathrm{D_{ufm}}$), a set J of "tight vertices" is defined as follows:

$$J = \left\{ u : \sum_{e \in E(u)} \pi_e = c_u \right\}$$

For the complementary slackness conditions to be satisfied, for each $j \notin J$, we must fix the corresponding x_j to be 0. Furthermore for each $j \in J$, the corresponding primal variable x_j can be set to an arbitrary value. The restricted primal ($\mathrm{RP_{ufm}}$) tells us how far we are from satisfying the constraints in the the primal.

$$\begin{array}{lll} \textit{minimize} & \sum_{e \in E} s_e & \\ \textit{subject to} & x_u + x_v + s_e = 1 & \forall (u,v) \in E \\ & s_e \geq 0 & \forall e \in E \\ & x_u \geq 0 & \forall u \in J \\ & x_u = 0 & \forall u \notin J \end{array} \qquad (\mathrm{RP_{ufm}})$$

where s_e is a slack variable for edge e.

The final part of the primal-dual algorithm requires us to look at the dual of the restricted primal ($\mathrm{DRP_{ufm}}$):

$$\begin{array}{lll} \textit{maximize} & \sum_{e \in E} \pi_e & \\ \textit{subject to} & \sum_{e \in E(u)} \pi_e \leq 0 & \forall u \in J \\ & \pi_e \gtrless 0 & \forall e \in E \\ & \pi_e \leq 1 & \forall e \in E \end{array} \qquad (\mathrm{DRP_{ufm}})$$

An optimal solution $\overline{\pi}$ for ($\mathrm{DRP_{ufm}}$) will allow us to find a multiplying factor θ with which $\overline{\pi}$ needs to be multiplied to improve the original solution π. The new dual solution π^* is now:

$$\pi^* = \pi + \theta \overline{\pi}$$

where

$$\theta = \min_{u \notin J,\ \overline{\pi}(u) > 0} \left\{ \frac{c_u - \sum_{e \in E(u)} \pi_e}{\sum_{e \in E(u)} \overline{\pi}_e} \right\}$$

After each iteration of the primal-dual algorithm, at least one vertex u will enter the set J while all others in J remain in the set. Hence, the problem of finding an optimal solution to the unconstrained fractional matching problem ($\mathrm{D_{ufm}}$) can be reduced to finding an optimal solution to an instance of ($\mathrm{DRP_{ufm}}$).

Next we present some notation:

Saturated and unsaturated vertices:
Given a feasible solution π to the dual problem, a vertex u is *unsaturated with respect to* π if

$$\sum_{e \in E(u)} \pi_e < c_u.$$

On the other hand, a *saturated vertex* u *with respect to* π has the property that

$$\sum_{e \in E(u)} \pi_e = c_u.$$

Saturated and unsaturated edges:
Given a feasible solution $\overline{\pi}$ to the DRP, an edge e *is unsaturated with respect to* $\overline{\pi}$ if $\overline{\pi}_e < 1$. If $\overline{\pi}_e = 1$, edge e is *saturated with respect to* $\overline{\pi}$.

Augmenting path (lasso):
An *augmenting path (lasso)* P $(u_1, e_1, u_2, e_2, ..., u_t, e_t, u_{t+1})$ of length t *with respect to a feasible solution* $\overline{\pi}$ is a sequence of vertices and edges such that each vertex u_i is connected to u_{i+1} through edge e_i in the graph G (for $i \in [t]$). An augmenting path (lasso) P has the following properties:

1. P is odd in length ($t = 2k + 1$ for some integer k)
2. P begins and ends at unsaturated vertices u_1 and u_{t+1} with respect to π.
3. All **intermediate vertices** $u_2, ..., u_t$ are saturated with respect to π.
4. All odd edges in P $(e_1, e_3, e_5...)$ are unsaturated with respect to $\overline{\pi}$. In other words, $\overline{\pi}_{e_{2i-1}} < 1$ $(i \in [\frac{t+1}{2}])$.

If vertices in P are pairwise distinct then P is an *augmenting path*; otherwise, it is an *augmenting lasso*.

The main result of this paper is a combinatorial algorithm for finding an optimal solution to the DRP of the unconstrained fractional matching problem. The proof uses the characterization in Theorem 1 whose proof is similar to the proof in [6].

Theorem 1 *(Correctness Theorem) Let* $\overline{\pi}$ *be a feasible solution to an instance of* ($\mathrm{DRP_{ufm}}$) *found by repeatedly finding augmenting paths/lassos and augmenting them starting at an initial solution of* $\overline{\pi} = 0$*. Then,* $\overline{\pi}$ *is optimal if and only if there does not exists an augmenting path/lasso with respect to* $(\pi, \overline{\pi})$ *in the graph* G*.*

Given a general graph $G(V,E)$, we first construct a bipartite graph $G^B(V_1,V_2,E^B)$ such that $V_1 = \{v_1 : v \in V\}$, $V_2 = \{v_2 : v \in V\}$ and $E^B = \{(u_1,v_2),(u_2,v_1) : (u,v) \in E\}$. Furthermore, define $c(u_1) = c(u_2) = c(u)$ for all $u_1(u_2) \in V_1(V_2)$. Given a general graph G, and its corresponding bipartite reduction G^B, the set of feasible assignments of edge values π^B to G^B has a one-to-one mapping onto the set of feasible assignments of edge values π to G. Constructing G^B from the general graph G eliminates any possibility of augmenting lassos. Here we omit the details of how to update the values on edges in G given the values on edges in G^B.

Next we derive an algorithm for solving an instance of ($\mathrm{DRP}_{\mathrm{ufm}}$). This algorithm works in phases, similar to the Hopcroft and Karp algorithm for bipartite matching [1]. Using alternating breadth first search on the bipartite graph $G^B(V_1,V_2,E^B)$ starting from the set of saturated vertices J^B we discover a maximal set of unsatured edge disjoint augmenting paths of the shortest length in a phase. All these paths are augmented simultaneously. If an augmenting path cannot be found, then by Theorem 1, we are at an optimal solution to ($\mathrm{DRP}_{\mathrm{ufm}}$). We omit the proofs but it can be shown that the number of phases is at most $O(\sqrt{|E|})$ and each phase takes at most $O(|E|)$ time. Also the total number of times DRP needs to be solved is $O(|V|)$. Therefore, the running time of the algorithm is $\mathrm{O}(|V||E|^{\frac{3}{2}})$; an improvement over other naïve implementations which run in $O(|V||E|^2)$ time.

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Global defensive alliances in double-loop networks

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We are acquainted with alliance by referring to [3] and [6]. A *defensive alliance* in graph $G = (V, E)$ is a set of vertices $S \subseteq V$ satisfying $|N[v] \cap S| \geq |N(v) \cap (V - S)|$ for any $v \in S$, $N(v) = \{u : uv \in E\}$, and $N[v] = N(v) \cup \{v\}$. Because of such an alliance, the vertices in S, agreeing to mutually support each other, have the strength of numbers to be able to defend themselves from the vertices in $V - S$. A defensive alliance S is called *global* if $N[S] = V$. The *global alliance number* $\gamma_a(G)$ is the minimum cardinality of an alliance of G that is also a dominating set of G. The graphs which has been studied with global alliance are K_n, $K_{r,s}$, P_n, C_n and $S_{r,s}$. In [3], $\gamma_a(K_n) = \lfloor \frac{n+1}{2} \rfloor$, $\gamma_a(K_{1,s}) = \lfloor \frac{s}{2} \rfloor + 1$, and $\gamma_a(K_{r,s}) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor$ if $r, s \geq 2$. Moreover, the global alliance problem is NP-complete for general graphs [2].

A *double-loop network* $\overrightarrow{DL}(n; a, b)$[7] can be viewed as a directed graph with n vertices $0, 1, 2, \ldots, (n-1)$ and $2n$ directed edges of the form $i \to i + a \pmod n$ and $i \to i + b \pmod n$, referred to as a-links and b-links. In this talk, any reference to $DL(n; a, b)$ will mean an underlying graph of a directed graph $\overrightarrow{DL}(n; a, b)$. Therefore, we have $DL(n; a, b) \cong DL(n; n-a, b) \cong DL(n; a, n-b) \cong DL(n; n-a, n-b)$.

Lemma 1 *If* $\gcd(n, a) = 1$ *or* $\gcd(n, b) = 1$*, then* $DL(n; a, b) \cong DL(n; 1, k)$ *for some* k.

Lemma 2 *If* $\gcd(n, a, b) = d$ *then* $DL(n; a, b)$ *is the disjoint union of* d *graphs which are all isomorphic to* $DL(\frac{n}{d}; \frac{a}{d}, \frac{b}{d})$.

By the definition of $\gamma_a(G)$, we have the fact that if G is the disjoint union of G_1 and G_2, then $\gamma_a(G) = \gamma_a(G_1) + \gamma_a(G_2)$. Lemma 1, 2 and the above fact lead us to study $\gamma_a(G)$ with $G = DL(n; 1, k)$. We determine the value of the global defensive alliance number in $DL(n; 1, 2)$, $DL(n; 1, 3)$, $DL(3n; 1, 3k)$, and $DL(n; 1, \lfloor \frac{n}{2} \rfloor)$. Finally, we research into the relation between $\gamma_a(G)$ and integer programming for G being a k-regular graph.

Theorem 3 $\gamma_a(DL(n; 1, 2)) = \begin{cases} \lceil \frac{3(n+1)}{7} \rceil, & n = 10 \text{ and } n \equiv 1 \text{ or } 2 \pmod 7; \\ 3\lceil \frac{n}{7} \rceil, & \text{otherwise.} \end{cases}$

Theorem 4 $\gamma_a(DL(n;1,3)) = \begin{cases} \lceil \frac{n}{3} \rceil, & n = 11 \text{ and } n \equiv 0 \text{ or } 1 \pmod 3; \\ \lceil \frac{n}{3} \rceil + 1, & \text{otherwise.} \end{cases}$

Theorem 5 $\gamma_a(DL(3n;1,3k)) = n.$

Theorem 6 $\gamma_a(DL(n;1,\lfloor \frac{n}{2} \rfloor)) = \begin{cases} \gamma_a(DL(n;1,2)), & n \text{ is odd;} \\ \lceil \frac{n}{3} \rceil + 1, & n = 6k+2 \text{ and } k \text{ is even;} \\ \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases}$

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Adjacent vertex distinguishing edge-colorings of planar graphs with girth at least six

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Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *proper k-edge-coloring* is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. Let $C_\phi(v) = \{\phi(xv) \mid xv \in E(G)\}$ denote the set of colors assigned to edges incident to the vertex v. A proper k-edge-coloring ϕ of G is *adjacent vertex distinguishing*, or a *k-avd-coloring*, if $C_\phi(u) \neq C_\phi(v)$ whenever u and v are adjacent vertices. The *adjacent vertex distinguishing chromatic index*, denoted $\chi'_a(G)$, is the smallest integer k such that G has a k-avd-coloring. Adjacent vertex distinguishing colorings are variously known as adjacent strong edge coloring [5] and 1-strong edge coloring [1]. Note that an isolated edge has no avd-coloring and a k-avd-coloring can be regarded as an m-avd-coloring for any $m \geq k$.

The *chromatic index* $\chi'(G)$ of a graph G is the smallest integer k such that G has a proper k-edge-coloring. Evidently, $\chi'_a(G) \geq \chi'(G)$. Let $\Delta(G)$ denote the maximum degree of G. The well-known Vizing Theorem [4] asserts that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ for every graph G. In contrast, there exist infinitely many graphs G such that $\chi'_a(G) > \Delta(G) + 1$. For instance, it is proved in [5] that, if $n \not\equiv 0 \pmod 3$ and $n \neq 5$, then the cycle C_n satisfies $\chi'_a(C_n) = 4 = \Delta(C_n)+2$. However, $\chi'_a(C_5) = 5 = \Delta(C_5)+3$.

Zhang, Liu, and Wang [5] completely determined the adjacent vertex distinguishing chromatic indices for paths, cycles, trees, complete graphs, and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

Conjecture 1. *If G is a connected graph with at least 6 vertices, then $\chi'_a(G) \leq \Delta(G) + 2$.*

Balister, Győri, Lehel, and Schelp [2] established the following three theorems.

Theorem 1 *If G is a graph without isolated edges and $\Delta(G) = 3$, then $\chi'_a(G) \leq 5$.*

Theorem 2. *If G is a bipartite graph without isolated edges, then $\chi'_a(G) \leq \Delta(G)+2$.*

Theorem 3. *If G is a graph without isolated edges and the chromatic number of G is k, then $\chi'_a(G) \leq \Delta(G) + O(\log k)$.*

The following bound proved by Hatami [3] is better than Theorem 3 for graphs with extremely large chromatic numbers.

Theorem 4. *If G is a graph without isolated edges and $\Delta(G) > 10^{20}$, then $\chi'_a(G) \leq \Delta(G) + 300$.*

In this paper, we prove Conjecture 1 for the following case.

Theorem 5. *If G is a plane graph without isolated edges whose girth $g(G) \geq 6$, then $\chi'_a(G) \leq \Delta(G) + 2$.*

Our proof proceeds by *reductio ad absurdum*. Assume that G is a counterexample to the theorem whose $|V(G)| + |E(G)|$ is the least possible. We are going to analyze the structure of G with a sequence of auxiliary lemmas. Then we will derive a contradiction using the discharging method.

Lemma 1. *No vertex of degree 2 is adjacent to a leaf.*

Lemma 2. *If the degree $d_G(v)$ of v in G is at least 3, then there are at least three neighbors of v that are not leaves.*

A path $x_0x_1 \cdots x_kx_{k+1}$ of length $k+1$ in G is called a *k-chain* if $d_G(x_0) \geq 3$, $d_G(x_{k+1}) \geq 3$, and $d_G(x_i) = 2$ for all $i = 1, 2, \ldots, k$.

Lemma 3. *There does not exist any k-chain if $k \geq 3$.*

Lemma 4. *There exists no edge xy with $d_G(x) = 2$ and $D_G(y) = 3$, where $D_G(y)$ denotes the number of neighbors of y that are not leaves.*

Lemma 5. *There does not exist a vertex v with neighbors $v_1, v_2, \ldots, v_k$, $k \geq 4$, such that $d_G(v_1) = d_G(v_2) = 2$, $d_G(v_3) \geq 2$, $d_G(v_4) \geq 2$, and $d_G(v_i) = 1$ for all $i = 5, 6, \ldots, k$.*

Lemma 6 *There does not exist a vertex v with neighbors $v_1, v_2, \ldots, v_k$, $k \geq 5$, such that $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$, $d_G(v_4) \geq 2$, $d_G(v_5) \geq 2$, and $d_G(v_i) = 1$ for all $i = 6, 7, \ldots, k$.*

Lemma 7. *There does not exist a face $f = [x_1x_2 \cdots x_6]$ such that $d_G(x_i) = 2$ for all x_i except x_1 and x_4.*

Let H be the graph obtained by removing all leaves of G. Then H is a connected proper subgraph of G. It follows from Lemmas 1 and 2 that, for every $v \in V(H)$, $d_H(v) \geq 2$ and $d_H(v) = d_G(v)$ if $2 \leq d_G(v) \leq 3$.

Using $\sum_{v \in V(H)} d_H(v) = \sum_{f \in F(H)} d_H(f) = 2|E(H)|$ and Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$, we can derive the following identity.

$$\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12.$$

We define a weight function w by $w(v) = 2d_H(v) - 6$ for $v \in V(H)$ and $w(f) = d_H(f) - 6$ for $f \in F(H)$. It follows from the above identity that the sum of all weights is equal to -12. We will design appropriate discharging rules and then redistribute weights accordingly. Once the discharging is finished, a new weight function w' is produced. The sum of all weights is kept fixed when the discharging is in progress. However, the outcome $w'(x)$ is nonnegative for all $x \in V(H) \cup F(H)$. This leads to the following obvious contradiction.

$$0 \leq \sum_{x \in V(H) \cup F(H)} w'(x) = \sum_{x \in V(H) \cup F(H)} w(x) = -12.$$

There are two discharging rules.

(R1) If v is a vertex of degree 2 incident to a face f, then f sends 1 to v for each occurrence of v in the boundary walk of f.

A face $f = [uvw \cdots]$ of H is called a *light face belonging to* v if $d_H(v) \geq 4$ and either $d_H(u) = 2$ or $d_H(w) = 2$.

(R2) If $d_H(v) \geq 4$, then v sends 1 to each light face belonging to v.

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Minimum statuses of connected graphs

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Suppose that G is a connected graph. For a vertex x in G, the *status* $s_G(x)$ of x is defined by

$$s_G(x) = \sum_{y \in V(G)} d_G(x, y),$$

where $d_G(x, y)$ denotes the distance between x and y. The *minimum status* $ms(G)$ of the graph G is defined by

$$ms(G) = \min_{x \in V(G)} s_G(x).$$

For a tree T with root z_0, the *height* $h(T)$ of T is defined by

$$h(T) = max_{x \in V(T)} d_T(z_0, x).$$

Let T be a rooted tree with root z_0 and $h(T) \geq 2$. If $deg_T(x) = k$ for every vertex x with $d_T(x, z_0) \leq h(T) - 2$, then T is called a *balanced k-tree*. If $deg_T(x) = k$ for every vertex x with $d_T(x, z_0) \leq h(T) - 1$, then T is called a *full k-tree*. A balanced k-tree of order n is denoted by $B_{n,k}$. Note that $B_{n,k}$ may not be unique if it is not full.

For integers $k \geq 2, h \geq 1$, let $N_{k,h}$ denote the number of vertices of a full k-tree with height h. Thus

$$N_{k,h} = 1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{h-1}.$$

We denote $ms(B_{n,k})$ by $b_{n,k}$. We have

$$b_{n,k} = k + 2k(k-1) + 3k(k-1)^2 + \cdots + (h-1)k(k-1)^{h-2} + h \cdot (n - N_{k,h-1}).$$

For integers $n > k \geq 2$, the *k-grass* $G_{n,k}$ is the graph with $V(G_{n,k}) = \{x_1, x_2, \cdots, x_n\}$ and $E(G_{n,k}) = \{x_i x_{i+1} : i = 1, 2, \cdots, n-k\} \cup \{x_{n-k+1} x_j : j = n-k+2, n-k+3, \cdots, n\}$.

We use $g_{n,k}$ to denote the value of $ms(G_{n,k})$. Thus

$$g_{n,k} = \begin{cases} \binom{\lfloor \frac{n}{2} \rfloor + 1}{2} + \binom{n - \lfloor \frac{n}{2} \rfloor - k + 1}{2} \\ +(k-1)(n - k - \lfloor \frac{n}{2} \rfloor + 1) & \text{if} \quad 2 \leq k \leq \lfloor \frac{n}{2} \rfloor, \\ \binom{n-k+1}{2} + k - 1 & \text{if} \quad \lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n-1. \end{cases}$$

Main result:

Theorem *Suppose that G is a connected graph of order n and $\Delta(G) = k \geq 2$. Then we have $b_{n,k} \leq ms(G) \leq g_{n,k}$. The first inequality becomes equality if and only if G contains some spanning $B_{n,k}$. If the second inequality is an equality, then G contains a spanning $G_{n,k}$.*

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$\bar{S}_k$-factorization of symmetric digraph of product graphs

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All the graphs considered here are finite. Let $G = (V, E)$ be a graph. Partition of G into edge - disjoint subgraphs G_1, G_2, ..., G_r such that $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_r)$ is called a *decomposition* of G and in this case we write $G = G_1 \oplus G_2 \oplus ... \oplus G_k$. Decomposition of the given graph into spanning subgraphs is called *factorization*. An r-regular spanning subgraph of G is called an *r-factor* of G. A cycle of length k is denoted by C_k. G^* is the symmetric digraph obtained from G by replacing every edge by a symmetric pair of arcs. Let $K^*_{n_1,n_2,...,n_m}$ denote the *symmetric complete multipartite digraph* with partite sets $V_1, V_2, ..., V_m$ of $n_1, n_2, ..., n_m$ vertices each. The *symmetric complete multipartite multi-digraph* $\lambda K^*_{n_1,n_2,...,n_m}$, is the symmetric complete multipartite digraph $K^*_{n_1,n_2,...,n_m}$ in which every arc is taken λ times. $\hat{S}_k$ is an $S_k = K_{1,k-1}$ (using vertices from two parts of the m-partite digraph)with arcs oriented from the center vertex to end vertices. An $\hat{S}_k-$factor is a spanning subgraph H of a directed graph G such that each component of H is an $\hat{S}_k$. Partition of G^* into arc-disjoint $\hat{S}_k-$factors is called an $\hat{S}_k-$factorization of G^*, denoted by $\hat{S}_k \parallel G^*$. $\bar{S}_k$ denote an *evenly partite star* in which all arcs are directed away from a center vertex to $k-1$ end vertices such that the center vertex is in V_i and $\frac{k-1}{2}$ end vertices in V_j and $\frac{k-1}{2}$ end vertices are in V_l, where $i \neq j \neq l$ and $\{i, j, l\} \in \{1, 2, ..., m\}$. An *$\bar{S}_k-$factor* is a spanning subgraph H of a directed graph G such that each component of H is an $\bar{S}_k$. Partition of G^* into arc-disjoint $\bar{S}_k-$factors is called an *$\bar{S}_k-$factorization* of G^*, denoted by $\bar{S}_k \parallel G^*$. The complete graph on m vertices is denoted by K_m and its complement is denoted by K^c_m.

For two graphs G and H, their *wreath product*, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) is joined to (g_2, h_2) by an edge whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. The *tensor product* of two graphs G and H, denoted by $G \times H$, has the vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. For graph theoretical terms which are not mentioned here see [1].

Nowadays, decomposition/factorization of product graphs into cycles, stars, paths etc. have attracted much attention due to their applications in many fields especially in G-designs. Study on factorization of digraphs into stars, paths and cycles are not new (see [2-10]). Ushio [5, 6, 7] has obtained a necessary and some sufficient conditions for the existence of an $\hat{S}_k-$factorization of complete bipartite and tripartite symmetric digraphs. Recently, the authors [2, 3] have obtained a necessary and some sufficient conditions for the existence of $\hat{S}_k-$factorization of $(K_m \times K_n)^*$ and $K^*_{n_1,n_2,\ldots,n_r}$.

Further, Ushio [9, 10] studied the existence of $\bar{S}_k-$factorizations of $K^*_{n_1,n_2,n_3}$ and $\lambda K^*_{n_1,n_2,n_3}$ and left open for $K^*_{n_1,n_2,\ldots,n_m}$ when $m > 3$.

In this paper, we have obtained a necessary and sufficient condition for the existence of an $\bar{S}_k-$factorization of $(C_m \circ \bar{K_n})^*$. Further some necessary and sufficient conditions are obtained for the existence of an $\bar{S}_k-$factorization in $(K_m \times K_n)^*$ and $K^*_{n_1,n_2,\ldots,n_m}$ and the results are stated below.

Theorem 1. *For odd $k \geq 3$, $(C_m \circ K_n^c)^*$ has an $\bar{S}_k-$factorization if and only if $n \equiv 0\,(mod\,\frac{k(k-1)}{2})$.*

Remark: Theorem 1 deduces a result of Ushio [9] for tripartite graphs.

Theorem 2. *If $(K_m \times K_n)^*$ has an $\bar{S}_k-$factorization, then $n \equiv k\,(mod\,\frac{k(k-1)}{2d})$ where $d = (m, k)$.*

Theorem 3. *For $m \geq 3$, $t \geq 1$, $\bar{S}_k \,\|\, (K_m \times K_{k^t})^*$.*

Theorem 4. *Let $k \geq 3$ be odd. If $K^*_{n_1,n_2,\ldots,n_m}$ has an $\bar{S}_k-$factorization, then*

(i) $n_1 = n_2 = \ldots = n_m$, for $k = 3$

(ii) $n_1 = n_2 = \ldots = n_m = n \equiv 0\,(mod\,\frac{k(k-1}{2d})$ for $d = (m, k), k \geq 5$

(iii) $(m-1)n \equiv 0\,(mod\,(k-1))$

(iv) $mn \equiv 0\,(mod\,k)$ and

(v) $(m-1)n \geq (k-1)(\frac{k-1}{2})$.

Theorem 5. *If m is odd and $n \equiv 0\,(mod\,\frac{k(k-1)}{2})$, then $\bar{S}_k \,\|\, (K_m \circ K_n^c)^*$.*

Theorem 6. *For $k \geq 3$, $\bar{S}_k \,\|\, (K_k \circ K^c_{\frac{k-1}{2}})^*$.*

Theorem 7. *If $n \equiv 0\,(mod\frac{k-1}{2})$and odd $m \equiv 0\,(mod\,k)$, then $\bar{S}_k \,\|\, (K_m \circ K_n^c)^*$.*

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Algorithmic issues in the intersection graphs of 2D objects and their applications

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Abstract

Geometric intersection graphs are intensively studied in the literature. A geometric intersection graph $G(V, E)$ is defined with a set of geometric objects. Usually, the vertices correspond to the geometric objects; an edge $e_{ij} \in E$ between a pair of nodes $v_i, v_j \in V$ implies that the objects corresponding to v_i and v_j intersect. The problems on geometric intersection graphs are studied for their interesting theoretical properties, and for the practical motivations. Many such graph classes allow elegant characterizations. Different graph-theoretic optimization problems, which are usually NP-hard for arbitrary graphs, can be solved in polynomial time for some geometric intersection graphs. There are many optimization problems which remain NP-hard for the geometric intersection graphs also. In this note, we present the recent algorithmic results for several geometric intersection graphs.

1 Introduction

Consider a set of objects distributed in $\mathbb{R}^d$. The geometric intersection graph $G = (V, E)$ with this set of objects is an undirected graph whose each node corresponds to a distinct object in this set. Between a pair of nodes $v_i, v_j \in V$ there is an edge e_{ij} $(\in E)$ if the corresponding objects intersect. The intersection of a pair of objects is defined depending on the problem specification. For example, sometimes proper containment is considered to be an intersection and sometimes it is not. Here two types of problems are usually considered: (i) characterization problems, and (ii) solving some useful optimization problems. In the characterization problem, given an arbitrary graph, one needs to check whether it belongs to the intersection graph of a desired type of objects. The second kind of problem deals with designing efficient algorithms for solving some useful optimization problems for an intersection graph of a known type of objects. It needs to be mentioned that several practically useful optimization problems, for example, finding the largest clique, minimum vertex cover, maximum independent set, etc. are NP-hard for general graph. There are some problems for which getting an efficient approximation algorithm with good approximation factor is also very difficult. In this area of research, the geometric properties of the intersecting objects are used to design efficient algorithms for these optimization problems. The characterization problem is important in the sense that for the intersection graph of some types of objects, efficient algorithms are sometimes already available for solving the desired optimization problem.

Let us first consider the *interval graph*, which is the simplest type of geometric intersection graph. This is obtained by the intersection of a set of intervals on a real line. The characterization problem for the interval graph can easily be solved in $O(|V|+|E|)$ time by showing that the graph is chordal and its complementary graph is a comparability graph [18]. All the standard graph-theoretic optimization problems, for example, finding minimum vertex cover, maximum independent set, largest clique, minimum clique cover, minimum coloring, etc, can be solved in polynomial time for the interval graph [18]. In the next two sections we shall consider the intersection graphs of rectangles and circles in 2D, which are the most common types of objects used for modeling many practical problems.

2 Rectangle intersection graph

Any graph $G = (V, E)$ can be represented as the intersection graph of a set of axis-parallel boxes in some dimension. The boxicity of a graph with n nodes is the minimum dimension d such that the given graph can be represented as an intersection graph of n axis parallel boxes in dimension d. In Figure 1, a graph with boxicity 2 is demonstrated. A graph has boxicity at most one if and only if it is an interval graph. Every outerplanar graph has boxicity at most two, and every planar graph has boxicity at most three [27]. If a bipartite graph has boxicity two, it can be represented as an intersection graph of axis-parallel line segments in the plane [6]. Given an arbitrary undirected graph G, testing whether the boxicity of G is a given constant k is NP-complete even if $k = 2$ [8].

Figure 1: Rectangle intersection graph.

Many optimization problems can be solved or approximated more efficiently for graphs with bounded boxicity which are in general NP-complete for other graphs. For instance, the maximum clique problem for the intersection graph of axis-parallel rectangles in 2D can be computed in $O(n \log n)$ time using a plane sweep strategy [23]. This algorithm can be used to solve the following proximity problem - *given a set of points in 2D and a rectangle of specified size, report the position of the rectangle on the plane such that it can contain the maximum/minimum number of points.* It may be noted that the number of maximal cliques in a rectangle intersection graph is $O(n^2)$.

We now discuss on the maximum independent set and minimum clique cover problem of the rectangle intersection graph in 2D. It can be shown that the minimum vertex cover and maximum independent set problems for an intersection graph G of axis-parallel rectangles are both NP-hard [10]. It remains NP-hard even if these rectangles are unit squares [9]. If G is the intersection graph of a set of axis-parallel rectangles of arbitrary size, then a simple divide and conquer algorithm exists for getting an independent set of G which can be at most $\log n$-factor worse than the maximum independent set of G [4]. But if the rectangles are of fixed width, we can design a dynamic programming based 2-factor approximation algorithm for the maximum independent set problem for G using the shifting strategy as follows [4].

> Let the rectangles are of width h, and the layout of the rectangles on the plane is given. Draw horizontal lines at equal interval h. Note that each rectangle on the plane will be stabbed by one and only one horizontal line. Now consider a horizontal line ℓ. Let it stab m rectangles, and the intervals created by these rectangles on the line ℓ form an interval graph with m nodes. As stated earlier, the maximum independent set of this interval graph can be computed in $O(m \log m)$ time. We consider the odd numbered lines and compute the maximum independent set of the rectangles stabbed by them, and then merge these independent sets to get an independent set of G. This is possible due to the fact that if we consider a pair of odd numbered horizontal lines ℓ_1 and ℓ_2, there exists no rectangle stabbed by both ℓ_1 and ℓ_2. Similarly, considering the set of even numbered lines also one can compute a maximum independent set of G. Now we choose one of these maximum independent sets which has larger cardinality.

Similar shifting strategy can be adopted to design a 2-factor approximation algorithm for the minimum vertex cover and minimum clique cover problems for the rectangle intersection graph where the rectangles are of fixed width [2].

Both the minimum clique cover and maximum independent set problems for the rectangle intersection graph of equal-width rectangles are shown to admit PTAS [2, 4]. In [4], a dynamic programming based algorithm is proposed to compute an independent set of a rectangle intersection graph, and it produces a $(1 + \frac{1}{\epsilon})$-approximation in $O(n \log n + n^{2\epsilon-1})$ time, where $\epsilon \geq 1$ is an integer. In [2], it is shown that for a set of n equal width rectangles and a constant ϵ, a $(1+\epsilon)$-factor approximation of the minimum clique cover can be obtained in $O(n^{1/\epsilon^2})$ time.

The maximum independent set of rectangle intersection graph is extensively used in the map labeling problem. For references, see [4, 25, 26]. Here each city in a map can be considered as a point. The objective is to label these cities by their names such that no two labels overlap. Obviously if the point set is dense, then all the points can not be labeled in a non-intersecting manner. The objective here is to label the maximum number of points. Since the character size of the labels are the same, we have a set of rectangles of varying length but fixed width corresponding to all the points (cities), and we need to compute the maximum independent set of the corresponding rectangle intersection graph. Several variations of the map labeling problem are studied depending on the position of the point on its corresponding label, for example, a specific corner, or any one of the four corners, or on the boundary, etc.

3 Disk graphs

A graph $G=(V,E)$ is said to be a disk graph if it is obtained from the intersection of a set of disks. Here nodes correspond to the disks and an edge between a pair of vertices in G implies that the corresponding disks intersect. If the disks are of same radius, then the corresponding graph is referred to as unit disk graph (see Figure 2). Recognizing whether an arbitrary (given) graph is a unit disk graph is NP-complete [7].

Unit disk graphs play important roles in formulating different important problems in mobile ad hoc network. In mobile network, the base stations can be viewed as nodes of an unit disk graph, where the range of each base station is the same. Different practical problems on this network can be formulated as the graph-theoretic problems on unit disk graph. We now discuss on some important optimization problems on unit disk graph.

Figure 2: Unit disk graph.

3.1 Maximum clique

Let us consider the cliques of the disk graph. Let $v_i, v_j, v_k \in V$ form a clique in G and C_i, C_j, C_k are the corresponding circles. Now, C_i, C_j, C_k may or may not have a common region (see Figure 3). In the former case it is called a geometric clique, and in the latter case it is called a graphical clique. Surely the geometric cliques are all graphical clique.

Figure 3: Cliques in disk graph.

Though the unit disks do not satisfy Helly property, the maximum (graphical) clique in an unit disk graph can be computed in polynomial time [11] provided the centers of the disks are given. If the geometric instance of the problem is not given then also the maximum clique problem can be solved with the same time complexity. However, for a disk graph of unequal size, the time complexity of finding the maximum clique is not known. It is proved that finding the largest clique in the intersection graph of ellipses is APX-hard. In other words, unless $P = NP$, there exists a constant θ such that there is no approximation algorithm with ratio better than θ. Hence there is no PTAS.

The geometric clique of an unit disk graph has an important significance. Suppose a set of points $P = \{p_1, p_2, \ldots, p_n\}$ on a 2D plane and an unit disk is given. The objective is to position the disk so that it can contain the maximum number of points. Finding an algorithm for this problem will be an interesting problem.

3.2 Minimum clique cover

In the minimum clique cover problem, the objective is to identify minimum number of cliques (graphical/geometric) that covers all the nodes in the graph. The graphical version is interesting in its own merit of theoretical interest. The problem is known to be NP-complete [12]. The following result is very important for designing approximation algorithms for the unit disk graph.

Result 1: *[5] If the centers of the disks lie inside a corridor bounded by a pair of parallel straight lines at a distance $\sqrt{3}$, then the graph becomes a cocomparability graph.*

Thus the minimum clique cover of a unit disk graph satisfying Result 1 can be found in $O(n^2)$ time by simply coloring its complimentary graph.

Given an arbitrary unit disk graph $G = (V, E)$, one can get a 3-approximation algorithm for the minimum clique cover problem as follows:

> Draw parallel lines such that the distance between each pair of consecutive lines is $\sqrt{3}$. Let $G_i = (V_i, E_i)$ be the subgraph of G with vertices corresponding to the circles with center in the i-th corridor. By Result 1, one can optimally compute the minimum clique cover Z_i of G_i. Now compute $Z = \cup Z_i$.

We now show that $|Z| \leq 3|Z^*|$, where Z^* is the minimum clique cover of G. Let C be a clique in Z^*, and $Z_i^* = C \cap V_i$. Since Z_i is the minimum clique cover of G_i, we have $|Z_i| \leq Z_i^*$. It is easy to follow that the centers of the disks associated to the vertices of C are contained in a region of the plane distributed along at most three strips. This implies that C is the union of at most three disjoint cliques belonging to Z_{j-1}^*, Z_j^* and Z_{j+1}^* for some j. That is, $|Z| = \sum |Z_i| \leq \sum |Z_i^*| \leq 3|Z^*|$.

Recently, Dumitrescu and Pach [16] proposed an $O(n^2)$ time randomized algorithm for the minimum clique cover problem with approximation ratio 2.16. They also proposed a polynomial time approximation scheme (PTAS) for this problem that runs in $O(1/\epsilon^2)$. It improves on a previous PTAS with $O(1/\epsilon^4)$ running time [24].

A different variation of this graph is the *penny graph* where no two unit disks do not intersect properly; here an edge between a pair of nodes imply that the corresponding two disks touch each other. The minimum clique cover problem remains NP-complete for the penny graph also. If the graph G is a penny graph, then the algorithm proposed in [12] produces a $\frac{3}{2}$-factor approximation result for the minimum clique cover problem.

The geometric version of the minimum clique cover problem for unit disk graph is important for the practical point of view. For a given set of points on a plane, it gives the minimum number of unit disks required to cover all the points. It has a lot of applications in wireless communication where the objective is to place the base stations to cover a set of radio terminals (sensors) distributed in a region. The problem is known to be NP-hard [14, 16]. Carmi et. al [14] proposed an approximation algorithm for this problem where the approximation factor is 38. In particular, if the points are distributed below a straight line L, and the base stations (of same range) are allowed to be installed on L then a 4-factor approximation algorithm can be obtained provided all the points lie within an unit distance from L. Consider another important variation of the geometric unit disk cover problem, called the homogeneous 2-hop broadcast problem in wireless communication. Here a message needs to be broadcasted from a source radio station to all the other radio stations in at most 2-hops. Each radio station can receive message, but if it needs to broadcast further, it needs extra power. The objective is to assign this extra power to minimum number of radio stations so that the 2-hop broadcast is possible. Two different optimization problems are studied in [15]. Given a set of radio stations and the source s, the objective of the first version is to find a range ρ so that the 2-hop broadcast from s is possible and the total cost is minimized, where the cost of a radio station is proportional to the square of its range. This problem can be solved in time $O(n^{2.376} \log n)$ by formulating the problem using matrix multiplication. In its second variation, the range of the base stations is ρ (given). It can easily be tested in $O(n \log n)$ time whether the 2-hop broadcast from s is possible with range ρ. If possible, the objective is to identify the minimum number of radio stations whom range ρ need to be assigned for successful 2-hop broadcast. An $O(n^2)$ time algorithm is proposed in [15] that computes a 2-factor approximation result for this problem. The h-hop broadcast range assignment problem ($h > 2$) is known to be NP-complete [13]. Here the range of a radio station can be any real number; the objective is to minimize the total cost. Ambuhl et al. [3] proposed a polynomial-time approximation scheme for the h-hop broadcast range assignment problem for a fixed $h > 2$. The proposed algorithm runs in $O(n^{\beta})$ time, where $\beta = O((8h^2/\epsilon)^{2^h})$. The same paper also demonstrated that for $h = 2$, the problem can be solved in polynomial time; the time complexity of the proposed algorithm is $O(n^7)$. Approximation algorithms are available for the unbounded broadcast problem (i.e., $h = n - 1$) in $\mathbb{R}^2$, with approximation factor equal to 6 [1].

3.3 Maximum independent set

The problem of computing the maximum independent set and minimum vertex cover for unit disk graph are known to be NP-complete [11]. Thus, the research on this topic is concentrated on designing the approximation algorithms. Most of the works on designing the approximation schemes for the unit disk graph assume that the geometric representation of the disks are given. A dynamic programming based shifting strategy was used by Erlebach et al. [17] to give a PTAS for the maximum weighted independent set in disk graph of arbitrary radius. Their proposed algorithm achieves approximation ratio $1-\epsilon$ in time $O(n^{O(1/\epsilon^2)})$ for a disk graph with n disks. If the centers of these disks lie inside a corridor (region bounded by a pair of parallel lines) of width k then their proposed algorithm produces optimum solution in $O(n^{4\lceil \frac{2k}{\sqrt{3}} \rceil})$ time. They also proposed an $(1-\frac{1}{r})$-factor approximation algorithm for the maximum independent set problem for the unit disk graph which runs in $O(rn^{4\lceil \frac{2(r-1)}{\sqrt{3}} \rceil})$ time. Jan van Leeuwen [19] introduced the concept of thickness to propose a fixed parameter tractable algorithm for finding the maximum independent set of an unit disk graph. An instance is said to have the thickness τ if the floor can be split into a set of strips of width 1 such that each strip contains at most τ disk centers. They show that an instance of the problem with thickness τ can be solved in $O(\tau^2 2^{2\tau} n)$

time. Though many PTAS is proposed for the maximum independent set problem for the unit disk graph, the best known constant factor approximation available for this problem achieves an approximation ratio 3 as follows :

Choose the left-most disk C. There can be at most 3 mutually non-intersecting disk that can intersect C. If we choose C in the independent set, then we can loose at most 3 disks in the independent set. After choosing C, we delete C and all the disks that overlap on C, and apply the same algorithm [22].

Recently an algorithm for computing a maximum independent set of an unit disk graph is proposed in [21] that can produce a solution of size at least $\frac{1}{2}OPT$ where OPT is the size of the optimum solution for a given instance of the problem. The time and space complexities of the proposed algorithm is $O(n^4)$.

The same paper also studies the maximum independent set problem for the *penny graph*. The problem is NP-hard [12]. An algorithm is proposed in [21] which can produce 2-factor approximation result for the coin graph in $O(n \log n)$ time.

4 Conclusion

A survey on the recent results on approximation algorithms for the minimum clique cover problem and maximum independent set problem of rectangle intersection graph and unit disk graph is presented. Several other optimization problems on these intersection graphs are also studied exhaustively. But intersection graphs of some other shapes, for example unequal radius disks or ellipses, etc. are not studied much. Another area interesting and useful area is the study of fixed parameter tractable algorithms for these optimization problems.

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C_k-Decompositions of complete equipartite graphs

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Let H_1, H_2,...,H_k be subgraphs of the graph G. H_1, H_2,...,H_k are said to *decompose* G if the edge set of G can be partitioned into E_1, E_2,...,E_k such that $G[E_i]$, $1 \le i \le k$, the subgraph induced by E_i, is isomorphic to H_i. If $H_i \cong H$, for each i, then we say that H *decomposes* G, and we denote this by $H|G$; in this case, we also say that G is decomposed into H. The complete m-partite graph in which each part has n vertices is denoted by $K_{m(n)}$.

The main problems in the decomposition K_n into cycles of length m were the following: (i) If n is odd and $m \le n$, then $C_m|K_n$ if and only if $m|\binom{n}{2}$, (ii) If n is even and $m \le n$, then $C_m|K_n - I$, where I is a perfect matching of K_n if and only if $m|(\binom{n}{2} - \frac{n}{2})$. These problems were settled by Alspach and Gavlas [1] and Šajna [10].

A C_k*-factorization* of $K_{m(n)}$, is a decomposition of it into 2-factors in which each component is a cycle of length k. Liu [6] proved the existence of a C_k-factorization of $K_{m(n)}$, with few exceptions, when the obvious necessary conditions are satisfied. However, the following problem is not yet settled.

Problem. For $3 \le k \le mn$, $C_k|K_{m(n)}$ if and only if $(m-1)n$ is even and $k|\binom{m}{2}n^2$.

This problem has been completely settled by Cavenagh [5] when the number of parts is 3.

Theorem 1. [5] *For $k \le 3n$, $C_k|K_{3(n)}$ if and only if k divides $3n^2$.*

However, the problem of decomposing a complete tripartite graph, $K_{r,s,t}$, into cycles of length 5 when the parts size are unequal seems to be notoriously difficult [9]. In [7], we completely characterize the existence of a C_p- decomposition of $K_{m(n)}$, where p is a prime and $5 \le p$.

Theorem 2. [7] *For any prime $p \ge 5$, $mn \ge p$ and $m \ge 3$, $C_p \,|\, K_{m(n)}$ if and only if $(m-1)n$ is even and $p \,|\, \binom{m}{2}n^2$.*

In [11], Smith obtained the following result.

Theorem 3. [11] *For any $m \ge 2$ and prime $p \ge 3$, the complete multipartite graph $K_{m(n)}$ (where $mn \ge 2p$) has a decomposition into cycles of length $2p$ if and only if*

$(m-1)n$ is even and $2p|\binom{m}{2}n^2$.

Recently, we have been informed [2] that Smith proved the conditions under which C_{3p}-decomposition of $K_{m(n)}$ and C_{p^2}-decomposition of $K_{m(n)}$ exist, where p is a prime. Very recently, Billington et al. [3, 4] proved that the obvious necessary conditions are sufficient for the existence of decompositions of $K_{5(n)}$ and $K_{4(n)}$ into cycles of length k.

Theorem 4. [3] *If $3 \leq k \leq 4n$, then $C_k|K_{4(n)}$ if and only if n is even and $k|6n^2$.*

Theorem 5. [4] *If $3 \leq k \leq 5n$, then $C_k|K_{5(n)}$ if and only if $k|10n^2$.*

We have also considered [8] the decomposition of the symmetric digraph $K^*_{m(n)}$ into directed cycles of length k. In this paper, for $k \leq m$, we prove that the obvious necessary conditions are sufficient for the existence of a C_k-decomposition of $K_{m(n)}$, where k is the product of distinct primes. Initially, we prove the following theorem.

Theorem 6. *Let $m \geq 3$ and let p and q be distinct primes with $pq \leq m$. Then $C_{pq} \,|\, K_{m(n)}$ if and only if $(m-1)n$ is even and $pq \,|\, m(m-1)n^2$.*

Using the above theorem, we complete the proof of the following theorem.

Theorem 7. *Let $m \geq 3$, and let $p_1, p_2, \ldots, p_k$, $k \geq 2$ be distinct primes with $p_1p_2\ldots p_k \leq m$. Then $C_r \,|\, K_{m(n)}$, where $r = p_1p_2\ldots p_k$, if and only if $(m-1)n$ is even and $r \,|\, m(m-1)n^2$.*

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b-coloring of graphs

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A k-coloring of a graph G is a b-coloring of G using k colors if each color class contains a color dominating vertex, that is, a vertex adjacent to at least one vertex of every other color class. The *b-chromatic number* of a graph G, denoted by $b(G)$, is the maximum k such that G has a b-coloring using k colors. Here also, as in the case of achromatic number, the chromatic number is the minimum k such that G has a b-coloring using k-colors. The b-chromatic number was introduced by R.W. Irving and D.F. Manlove [2] by considering proper colorings that are minimal with respect to a partial order defined on the set of all partitions of $V(G)$. They have shown that the determination of $b(G)$ is NP-hard for general graphs, but polynomial for trees. The name b-chromatic number is analogous to the achromatic number. While the achromatic number of a graph G gives the maximum number of color classes in a 'complete' coloring of G, the b-chromatic number of a graph G gives the maximum number of color classes in a b-coloring of G.

One of our aims in the study of b-coloring is to determine graphs with prescribed ω(= the clique number), χ (= the chromatic number), b (= the b-chromatic number) and Δ (= the maximum degree).

In the mid 20^{th} century there was a question regarding the construction of triangle-free k-chromatic graphs for $k \geq 3$. In this search, Mycielski [6] developed an interesting graph transformation known as the *Mycielskian* as follows. For a graph $G = (V, E)$, the Mycielskain of G is the graph $\mu(G)$ whose vertex set is the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{xy' \ : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex x (and x the twin of x') and the vertex u is called the root of $\mu(G)$. For $k \geq 2$, $\mu^k(G)$ is defined iteratively by setting $\mu^k(G) = \mu(\mu^{k-1}(G))$.

The Mycielskian $\mu(G)$ of a graph has the property that $\chi(\mu(G)) = \chi(G) + 1$ and $\omega(\mu(G)) = \omega(G)$. Hence the iterated Mycielskian gives graphs with prescribed χ and ω. Naturally, we thought this contribution could be used to produce graphs satisfying our aim. In this process, we determine bounds for the b-chromatic numbers of Mycielskian and generalised Mycielskain (a generalisation of the concept of Mycielskian), of a graph in terms of the b-chromatic number of the original graph. In particular, we have shown that if G is a graph with b-chromatic number b and for which the number of vertices of degree at least b is at most $2b - 2$, then $b(\mu(G))$ belongs to the interval

$[b+1, 2b-1]$. As a consequence, it follows that $b(G)+1 \leq b(\mu(G)) \leq 2b(G)-1$ for G in any of the following families: split graphs, $K_{n,n} - \{\text{a 1-factor}\}$, the hypercubes Q_p, where $p \geq 3$, graphs with $b(G) = \Delta(G)+1$, trees and a special class of bipartite graphs. We have further shown that for any positive integer b and every integer $k \in [b+1, 2b-1]$, there exists a graph G belonging to the family mentioned above, with $b(G) = b$ and $b(\mu(G)) = k$. Also we have shown that for any integer $n \geq 2$ and for any integer $m \in [3, 2n-1]$, $\left\lfloor \frac{2mn}{m+1} \right\rfloor \leq b(\mu_m(K_n)) \leq 2n-1$ and that these bounds are optimal.

Let v be any vertex of a graph G. We know that for the chromatic number of the vertex-deleted subgraph $G-v$ of G, $\chi(G-v) = \chi(G)$ or $\chi(G)-1$. Similarly for the achromatic number $\psi(G)$, $\psi(G-v) = \psi(G)$ or $\psi(G)-1$. Surprisingly, a similar statement does not hold good for the b-chromatic number $b(G)$ of G. Indeed, the gap between $b(G-v)$ and $b(G)$ can be arbitrarily large.

Here we have established that for any connected graph G on $n \geq 5$ vertices and any $v \in V(G)$, $b(G) - \left(\left\lceil \frac{n}{2} \right\rceil - 2\right) \leq b(G-v) \leq b(G) + \left(\left\lfloor \frac{n}{2} \right\rfloor - 2\right)$. Further we have determined all the graphs which attain these bounds [1].

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Strongly and co-strongly perfect graphs

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A graph is perfect if for each of its induced subgraohs H, the chromatic number of H is equal to the maximum number of mutually adjacent vertices in H. The perfect graphs were discovered by Claude Berge during the early 1960s. In 1961 Berge conjectured that a graph is perfect if it does not contain an odd cycle ofr length at least 5 or its complement as an induced sub-graph. The conjecture was well known as "Strong Perfect Graphs Conjecture" and remained unsolved for more than four decades. In May 2002, the SPGC was settled in affirmative by Maria Chudnovsky, Neil Robertson, Paul Seymour and Robin Thomas. Earlier the conjecture was proved for many classes of graphs: $K_{1,3}$-free graphs (Parthasarathy and Ravindra, 1976), $(K_4 - e)$-free graphs (Parthasarathy and Ravindra, 1979, Tucker, 1987), K_4-free graphs (Tucker, 1984), Pan free graphs (Olariu, 1989), Bull free graphs (Chvatal and Sbihi, 1989), Dartfree graphs (L- Sun, 1991).

During the period from the Strong Perfect Graph Conjecture (SPGC) to Strong Perfect Graph Theorem (SPGT) for more than four decades several researchers worked actively in the area of perfect graphs. Even after SPGC to SPGT there is still a lot to do in perfect graphs.

There are some interesting classes of perfect graphs viz., strongly perfect graphs and co-strongly perfect graphs which are not characterized completely though some sufficient conditions are available. A graph is strongly perfect if each of its induced sub-graphs H contain and independent set which meets all the maximal complete subgraphs of H. A graph G is Co-strongly perfect if G and its complement are strongly perfect. Meyniel graphs, special $K_{1,3}$-free graphs (Line graphs), comparability graphs, perfectly orderable graphs, co-strongly perfect graphs are some of the most important classes of strongly perfect graphs. Here we discuss the results concerning strongly perfect graphs and costrongly perfect graphs.

Orthogonal double covers of graphs and mutually orthogonal graph squares

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A *decomposition* $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ of a graph G is a partition of $E(G)$, the edge set of G, into edge-disjoint subgraphs $G_1, G_2, \ldots, G_s$, called *pages*. If all the pages of $\mathcal{G}$ are isomorphic to H_0, then $\mathcal{G}$ is called a *decomposition of* G *by* H_0.

Let H be any graph and let $\mathcal{G} = \{G_1, G_2, \ldots, G_{|V(H)|}\}$ be a collection of $|V(H)|$ subgraphs of H. $\mathcal{G}$ is a *double cover* (DC) of H if every edge of H is contained in exactly two members in $\mathcal{G}$. (In other words, $\mathcal{G}$ is a decomposition of $H^{(2)}$, the graph obtained from H by replacing each edge uv of H by a pair of edges joining u and v.) If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{G}$ is a *DC* of H by G. (If $\mathcal{G}$ is a DC of H by G, then $|V(H)||E(G)| = 2|E(H)|$.)

A DC $\mathcal{G}$ of H is an *orthogonal double cover* (ODC) of H if there exists a bijective mapping $\phi : V(H) \to \mathcal{G}$ such that for every choice of distinct vertices u and v in $V(H)$, $|E(\phi(u)) \cap E(\phi(v))|$ is 1 if $uv \in E(H)$ and is 0 otherwise. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{G}$ is an *ODC* of H by G. (For every $n = \binom{k}{2} + 1$, an *ODC* of the complete graph K_n by the complete graph K_k corresponds to a symmetric $(n, k, 2)$ block design.)

An *automorphism* of an ODC $\mathcal{G}$ of H is a permutation $\pi : V(H) \to V(H)$ such that $\{\pi(G_1), \pi(G_2), \ldots, \pi(G_{|V(H)|})\} = \mathcal{G}$, where for $i \in \{1, 2, \ldots, |V(H)|\}$, $\pi(G_i)$ is a subgraph of H with $V(\pi(G_i)) = \{\pi(v) : v \in V(G_i)\}$ and $E(\pi(G_i)) = \{\pi(u)\pi(v) : uv \in E(G_i)\}$. An ODC $\mathcal{G}$ of H is *cyclic* (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$, the set of all automorphisms of $\mathcal{G}$.

Two decompositions $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ and $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ of the complete bipartite graph $K_{n,n}$ are said to be *orthogonal* if $|E(G_i) \cap E(F_j)| = 1$ for all $i, j \in \{1, 2, \ldots, n\}$. For orthogonality it is necessary that $|E(G_i)| = n = |E(F_i)|$ for all $i \in \{1, 2, \ldots, n\}$. (If two decompositions $\mathcal{G}$ and $\mathcal{F}$ of $K_{n,n}$ by G are orthogonal, then $\mathcal{G} \cup \mathcal{F}$ is an ODC of $K_{n,n}$ by G.)

A set of decompositions $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k\}$ of $K_{n,n}$ is called a set of k *mutually orthogonal graph squares* (MOGS) if $\mathcal{G}_i$ and $\mathcal{G}_j$ are orthogonal for all $i, j \in \{1, 2, \ldots, k\}$ and

$i \neq j$. For any bipartite graph G with n edges, $N(n, G)$ denotes the maximum number k in a largest possible set $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k\}$ of MOGS of $K_{n,n}$ by G. (A decomposition of $K_{n,n}$ by nK_2 is equivalent to a Latin square of side n. If two decompositions $\mathcal{G}$ and $\mathcal{F}$ of $K_{n,n}$ by nK_2 are orthogonal, then the corresponding Latin squares of side n are orthogonal, and conversely. $N(n, nK_2) = N(n)$ is the maximum number of Latin squares in the largest possible set of mutually orthogonal Latin squares (MOLS) of side n.)

Orthogonal double covers of graphs, Orthogonal decompositions of graphs and Maximum number of mutually orthogonal graph squares have been studied by several authors, see the survey articles [1], [2], [3] and [4], also see [5].

In this talk, we present an overview on the current state of research along with some new results and open problems on orthogonal double covers of graphs, orthogonal decompositions of $K_{n,n}$, and $N(n, G)$.

Few of our recent results in these highly regular decompositions are:

Theorem 1. ([10]) There exists a CODC of $K_{2r+2s+rs+1}$ by $K_{2,r,s}$.

Theorem 2. ([10]) There exists a CODC of $K_{2r+2s+rs+2}$ by $K_{1,1,r,s}$.

Theorem 3. ([10]) There exists a CODC of K_{4r+7} by $K_{1,1,1,1,r}$.

Theorem 4. ([9]) If p is an odd prime, then $N(p, P_{p+1}) = p$.

Theorem 5. ([11]) If $p \geq 5$ is a prime number, then $p-2 \leq N(p,\, P_4 + (p-3)K_2)$.

Theorem 6. ([11]) If $p \geq 5$ is a prime number, then $p-3 \leq N(p,\, 2P_3 + (p-4)K_2)$.

Theorem 7. ([11]) If $p \geq 7$ is a prime number, then $p-4 \leq N(p,\, 2P_4 + (p-6)K_2)$.

Theorem 8. ([11]) If $p \geq 7$ is a prime number, then $p-5 \leq N(p,\, 3P_3 + (p-6)K_2)$.

[6], [7] and [8] deals with CODCs of graphs by R. Sampathkumar and others.

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A simple linear time algorithm for constructing spanners in weighted graphs

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1 Introduction

A spanner is a *sparse* subgraph of a given graph that preserves approximate distance between each pair of vertices. More precisely, a t-spanner of a graph $G = (V, E)$ is a subgraph $(V, E_S), E_S \subseteq E$ such that, for any pair of vertices, their distance in the subgraph is at most t times their distance in the original graph. where t is called the *stretch factor*. The spanners were defined formally by Peleg and Schäffer [14] though the associated notion was used implicitly by Awerbuch [3] in the context of network synchronizers.

The computation a t-spanner is clearly motivated by the compression in size, we are interested in the smallest t spanner of any given graph. The *smallest* size (in number of edges) of a t-spanner depends on the input graph. For example, it is easy to observe that a complete unweighted graph has a 2-spanner (star) of size $n - 1$ whereas any bipartite graph has no 2-spanner except the graph itself. In general, computing a t-spanner of the smallest size for a graph is NP-hard. In fact, for $t > 2$, it is NP-hard [10] even to approximate the smallest size of t-spanner of a graph with ratio $O(2^{(1-\mu)\ln n})$ for any $\mu > 0$. Let $\mathcal{S}_n^t$ denote the maximum size of the sparsest t-spanner over all graphs on n vertices. Given any graph, we would like to design an efficient algorithm for computing a t-spanner of size $O(\mathcal{S}_n^t)$. A 43 years old girth lower bound conjecture by Erdös [12] implies that there are graphs on n vertices whose $2k$ as well as $(2k - 1)$-spanner will require $\Omega(n^{1+1/k})$ edges. This conjecture has been proved for $k = 1, 2, 3$ and 5. Our objective is to design a fast, preferably linear time algorithm for any weighted graph on n vertices to compute a $(2k - 1)$-spanner of $O(n^{1+1/k})$ size.

Althöfer et al. [2] are the first to design a polynomial time algorithm for computing a $(2k - 1)$-spanner of $O(n^{1+1/k})$ size for any weighted graph that requires $O(\min(kn^{2+1/k}, mn^{1+1/k}))$ time. Cohen [9], and later Thorup and Zwick [18] designed algorithms with an improved running time of $O(kmn^{1+1/k})$. These algorithms relied on several calls to Dijkstra's single-source shortest-path algorithm for global distance computation and therefore were far from achieving linear time.

2 Our results

The key result of this article is an algorithm [6] that computes a spanner without any sort of local/global distance computation, and runs in expected linear time. The main result can be stated formally as follows :

> Given a weighted graph $G = (V, E)$, and integer $k > 1$, a spanner of $(2k-1)$-stretch and $O(kn^{1+1/k})$ size can be computed in expected $O(km)$ time.

The algorithm executes in $O(k)$ rounds, and in each round it essentially explores adjacency list of each vertex to prune dispensable edges. As a testimony of its simplicity, we will present the entire algorithm for 3-spanner and its analysis in the next section. The local approach can be exploited to achieve near-optimal performance in other computational models. More specifically, it takes $O(k)$ rounds and communications of $O(km)$ messages to construct a $(2k-1)$-spanner in synchronous distributed environment. In external-memory, the expected number of I/O operations required to compute a $(2k-1)$-spanner is the same as required for sorting km integers in external memory. In CRCW PRAM model, a $(2k-1)$-spanner can be computed with optimal speed-up in $O(k \log^* n)$ steps. The expected size of the $(2k-1)$-spanner computed is $O(kn^{1+1/k})$ in all these models.

Suppose there is a subset $E_S \subset E$ that ensures the following proposition for every edge $(x, y) \in E \backslash E_S$.

$\mathcal{P}_t(x, y)$: the vertices x and y are connected in the subgraph (V, E_S) by a path consisting of at most t edges, and the weight of each edge on this path is not more than that of the edge (x, y).

It follows easily that the sub graph (V, E_S) will be a t-spanner of G.

Althöfer *et al.* [2] gave the first algorithm for computing a t-spanner for weighted graphs. Their algorithm is similar to Kruskal's algorithm for computing a minimum spanning tree. The edges of the graph are processed in the increasing order of their weights. To begin with, the spanner $E_S = \emptyset$ and the algorithm adds edges to it gradually. The decision as to whether an edge, say (u, v) has to be added (or not) to E_S is made as follows:

If the distance between u and v in the subgraph induced by the current spanner edges E_S is more than $t \cdot weight(u, v)$, then select and add the edge to E_S, otherwise discard the edge.

It follows that $\mathcal{P}_t(x, y)$ would hold for each edge missing in E_S, and so at the end of the process, the subgraph (V, E_S) will be a t-spanner. Moreover, the girth of the graph (V, E_S) is at least $t + 1$. It follows from elementary graph theory that a graph with more than $n^{1+1/k}$ edges must have a cycle of at most $2k$ edges. Hence for $t = 2k - 1$, the above algorithm computes a $(2k - 1)$-spanner of size $O(n^{1+1/k})$, which is indeed optimal based on the lower bound mentioned earlier. A simple $O(mn^{1+1/k})$ implementation of the algorithm follows by using Dijkstra's algorithm to compute shortest paths.

Since a spanner must approximate all pairs of distances in a graph, it appears difficult to compute a spanner by avoiding explicit distance information. Somewhat surprisingly, the linear time algorithm computes such set E_S using a very local approach.

2.1 Computing a 3-spanner in linear time

To meet the size constraint of a 3-spanner a vertex, on an average contributes $\sqrt{n}$ edges to the spanner. So the vertices with degree $O(\sqrt{n})$ are easy to handle since all their edges can be selected in the spanner. For vertices with higher degree a clustering (groupings) scheme is employed to tackle this problem which has its basis in *dominating sets*.

To begin with, there is a set of edges E' initialized to E, and empty spanner E_S. The algorithm processes the edges E', moves some of them to the spanner E_S and discards the remaining ones. It does so in the following two phases.

1. *Forming the clusters* :
   A sample $\mathcal{R} \subset V$ is chosen by picking each vertex independently with probability $\frac{1}{\sqrt{n}}$. The clusters will be formed around these sampled vertices. Initially the clusters are $\{\{u\} | u \in \mathcal{R}\}$. Each $u \in \mathcal{R}$ is called the *center* of its cluster. Each unsampled vertex $v \in V - \mathcal{R}$ is processed as follows.

   (a) If v is not adjacent to any sampled vertex, then every edge incident on v is moved to E_S.

   (b) If v is adjacent to one or more sampled vertices, let $\mathcal{N}(v, \mathcal{R})$ be the sampled neighbor that is nearest [1] to v. The edge $(v, \mathcal{N}(v, \mathcal{R}))$ along with every edge that is incident on v with weight less than this edge is moved to E_S. The vertex v is added to the cluster centered at $\mathcal{N}(v, \mathcal{R})$.

   As a last step of the first phase, all those edges (u, v) from E' where u and v are not sampled and belong to the same cluster are discarded.

[1] Ties can be broken arbitrarily. However, it helps conceptually to assume that all weights are distinct

Let V' be the set of vertices corresponding to the endpoints of the edges E' left after the first phase. It follows that each vertex from V' is either a sampled vertex or adjacent to some sampled vertex, and the step 1(b) has partitioned V' into disjoint clusters each centered around some sampled vertex. Also note that, as a consequence of the last step, each edge of the set E' is an inter-cluster edge. The graph (V', E'), and the corresponding clustering of V' is passed onto the second phase.

2. *Joining vertices with their neighboring clusters* :
   Each vertex v of graph (V', E') is processed as follows. Let $E'(v, c)$ be the edges from the set E' incident on v from a cluster c. For each cluster c incident to v, the least-weight edge from $E'(v, c)$ is moved to E_S and the remaining edges are discarded.

The number of edges added to the spanner E_S during the algorithm described above can be bounded as follows. Note that the sample set $\mathcal{R}$ is formed by picking each vertex randomly independently with probability $\frac{1}{\sqrt{n}}$. It thus follows from elementary probability that for each vertex $v \in V$, the expected number of incident edges with weight less than that of $(v, \mathcal{N}(v, \mathcal{R}))$ is at most $\sqrt{n}$. Thus the expected number of edges contributed to the spanner by each vertex in the first phase of the algorithm is at most $\sqrt{n}$. The number of edges added to the spanner in the second phase is $O(n|\mathcal{R}|)$. Since the expected size of the sample $\mathcal{R}$ is $\sqrt{n}$, therefore, the expected number of edges added to the spanner in the second phase is at most $n^{3/2}$. Hence the expected size of the spanner E_S at the end of the algorithm described above is at most $2n^{3/2}$. The algorithm is repeated if the size of the spanner exceeds $3n^{3/2}$. It follows using Markov's inequality that the expected number of such repetitions will be $O(1)$.

We now establish that E_S is a 3-spanner. Note that for every edge $(u, v) \notin E_S$, the vertices u, v belong to some cluster in the first phase. There are two cases now.
Case 1 : *(u and v belong to same cluster)*

Let u and v belong to the cluster centered at $x \in \mathcal{R}$. It follows from the first phase of the algorithm that there is a 2-edge path $u - x - v$ in the spanner with each edge not heavier than the edge (u, v). (This provides a justification for discarding all intra-cluster edges at the end of first phase).
Case 2 : *(u and v belong to different clusters)*
Clearly the edge (u, v) was removed from E' during phase 2, and suppose it was removed while processing the vertex u. Let v belong to the cluster centered at $x \in \mathcal{R}$

In the beginning of the second phase let $(u, v') \in E'$ be the least weight edge among all the edges incident on u from the vertices of the cluster centered at x. So it must be that $weight(u, v') \leq weight(u, v)$. The processing of vertex u during the second phase of our algorithm ensures that the edge (u, v') gets added to E_S. Hence there is a path $\Pi_{uv} = u - v' - x - v$ between u and v in the spanner E_S, and its weight can be bounded as $weight(\Pi_{uv}) = weight(u, v') + weight(v', x) + weight(x, v)$. Since (v', x) and (v, x) were chosen in the first phase, it follows that $weight(v', x) \leq weight(u, v')$ and $weight(x, v) \leq weight(u, v)$. It follows that the spanner (V, E_S) has stretch 3. Moreover both phases of the algorithm can be executed in $O(m)$ time using elementary data structures and bucket sorting.

The algorithm for computing a $(2k-1)$-spanner executes k iterations where each iteration is similar to the first phase of the 3-spanner algorithm. The ith iteration begins with a clustering $\mathcal{C}_i$ (for the first iteration the clustering is $\mathcal{C}_1 = \{\{v\} | v \in V\}$). Each cluster from $\mathcal{C}_i$ is sampled independently with probability $n^{-1/k}$. The vertices in unsampled clusters hook on to the nearest sampled clusters and the surviving clusters become *fatter* in every iteration. The number of clusters reduce by a factor of $n^{1/k}$ and after $k-1$ iterations, there are expected $n^{1/k}$ clusters left. For more details and formal proof, the reader may refer to [6].

2.2 Other related work

The notion of a spanner has been generalized in the past by many researchers.

Additive spanners : A t-spanner as defined above approximates pairwise distances with multiplicative error, and can be called a multiplicative spanner. In an analogous manner, one can define spanners that approximate pairwise distances with additive error. Such a spanner is called an additive spanner and the corresponding error is called *surplus*. Aingworth et al. [1] presented the first additive spanner of size $O(n^{3/2} \log n)$ with surplus 2. Baswana et al. [7] presented a construction of $O(n^{4/3})$ size additive spanner with surplus 6. It is a major open problem if there exists any sparser additive spanner.

(α, β)*-spanner* : Elkin and Peleg [11] introduced the notion of (α, β)-spanner for unweighted graphs, which can be viewed as a hybrid of multiplicative and additive spanners. An (α, β)-spanner is a subgraph such that the distance between any pair of vertices $u, v \in V$ in this subgraph is bounded by $\alpha\delta(u, v) + \beta$, where $\delta(u, v)$ is the distance between u and v in the original graph. Elkin and Peleg showed that an $(1 + \epsilon, \beta)$-spanner of size $O(\beta n^{1+\delta})$, for arbitrarily small $\epsilon, \delta > 0$, can be computed at the expense of sufficiently large surplus β. Recently Thorup and Zwick [19] introduced a spanner where the additive error is sublinear in terms of the *distance* being approximated.

Other interesting variants of spanner include *distance preserver* proposed by Bollobás et al. [8] and *Light-weight* spanner proposed by Awerbuch et al. [4]. A subgraph is said to be a d-preserver if it preserves exact distances for each pair of vertices which are separated by distance at least d. A light-weight spanner tries to minimize the number of edges as well as the total edge weight. A *lightness* parameter is defined for a subgraph as the ratio of total weight of all its edges and the weight of the minimum spanning tree of the graph. Awerbuch et al. [4] showed that for any weighted graph and integer $k > 1$, there exists a polynomially constructible $O(k)$-spanner with $O(k\rho n^{1+1/k})$ edges and $O(k\rho n^{1/k})$ lightness, where $\rho = \log(Diameter)$.

In addition to the above work on the generalization of spanners, a lot of work has also been done on computing spanners for special classes of graphs, e.g., chordal graphs, unweighted graphs, and Euclidean graphs. For chordal graphs, Peleg and Schäffer [14] designed an algorithm that computes a 2-spanner of size $O(n^{3/2})$, and a 3-spanner of size $O(n \log n)$. For unweighted graphs Halperin and Zwick [13] gave an $O(m)$ time algorithm for this problem. Salowe [17] presented an algorithm for computing a $(1 + \epsilon)$-spanner of a d-dimensional complete Euclidean graph in $O(n \log n + \frac{n}{\epsilon^d})$ time. However, none of the algorithms for these special classes of graphs seem to extend to general weighted undirected graphs.

3 Applications and open problems

Spanners are quite useful in various applications in the area of distributed systems and communication networks. In these applications, spanners appear as the underlying graph structure. In order to build compact routing tables [16], many existing routing schemes use the edges of a sparse spanner for routing messages. In distributed systems, spanners play an important role in designing *synchronizers*. Awerbuch [3], and Peleg and Ullman [15] showed that the quality of a spanner (in terms of stretch factor and the number of spanner edges) is very closely related to the time and communication complexity of any synchronizer for the network. The spanners have also been used implicitly in a number of algorithms for computing all pairs approximate shortest paths [5, 9, 18]. For a number of other applications, please refer to the papers [2, 3, 14, 16]. The running time as well as the size of the $(2k - 1)$-spanner computed by the algorithm described above are away from their respective worst case lower bounds by a factor of k. For any constant value of k, both these parameters are optimal. However, for the extreme value of k, that is, for $k = \log n$, there is deviation by a factor of $\log n$. Is it possible to get rid of this multiplicative factor of k from the running time of the algorithm and/or the size of the $(2k - 1)$-spanner computed ? It seems that a more careful analysis coupled with advanced probabilistic tools might be useful in this direction.

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Duality in graphs and logical topology survivability in layered networks

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Graphs provide natural and convenient representations of communication networks. Graph models and flow formulations have been extensively used in the study of a large number of issues relating to network analysis and design. Structural characteristics such as connectivity/fault survivability (a measure of robustness of networks when link or node failures occur) and diameter (a measure of delay incurred in transmission of information from one node to another) have served as parameters in the design of communication networks. There is a rich body of literature dealing with these and related issues.

There are two approaches to deliver telecommunication services. The first is to design a new system from scratch every time a new type or level of service is to be delivered. The second is to accomplish the desired goal using the resources available in the existing systems. The latter approach involves the concept of layering. Modern communication networks are designed using the layered approach. Typically, design using the layered approach involves mapping a guest graph (called a *logical topology*) onto a host graph (called the *physical topology*) satisfying certain requirements. For example, in *Wavelength Division Multiplexing* (WDM) based networks the physical topology is defined by a set of nodes and optical fibers connecting the nodes, and the logical topology is defined by a subset of nodes (for example, IP routers) and the lightpaths connecting these nodes in the physical topology. Thus each logical link represents a lightpath in the physical topology. This talk is concerned with survivability of the logical topology when physical nodes/links fail, leading to cascading failures at the logical layer.

The *Survivable Logical Topology Mapping* (SLTM) problem in an IP-over-WDM optical network is to map each link (u, v) in the logical topology (at the IP layer) into a lightpath between the nodes u and v in the physical topology (at the optical layer) such that failure of a physical link does not cause the logical topology to become disconnected. It is assumed that both the physical and logical topologies are 2-edge connected (in short, two-connected).

The problem of finding pair-wise (mutually) disjoint paths plays an important role in finding survivable mappings. This problem is well studied and is NP-complete in general [1]. However, it is possible to find pair-wise disjoint paths in some special cases e.g. when the topology is undirected three edge-connected and the number of pair-wise disjoint paths required is two [2].

In [3], Modiano and Narula-Tam formally showed that the problem of finding survivable mappings is NP-complete for general as well as for ring logical topologies. Therefore, they provided *Integer Linear Programs* (ILPs) to find a solution. The ILP is based on the observation that a logical topology can get disconnected after the failure of a physical link only if the physical link carries an entire cut of the logical topology, or alternatively, every cut of the logical topology must contain at least a pair of edges with pair-wise disjoint mappings in order for the mappings to be survivable. However, the ILP does not scale well as it must examine all the possible cuts, a number that grow exponentially with the size of the topology. Some of the other related works on this problem may be found in [4, 5].

In [6, 7, 8] Kurant and Thiran provided a framework called SMART (*Survivable Mapping Algorithm by Ring Trimming*). SMART utilizes circuits to find survivable mappings for logical topologies. The framework repeatedly picks connected pieces (subgraphs) of the logical topology and finds survivable mappings for these pieces. If a survivable mapping is found for a piece, its links are short-circuited (contracted) and the algorithm proceeds by picking another piece. The process is repeated until the logical topology is reduced to a single node or a search for a piece with survivable mapping is unsuccessful. In the former case, a survivable mapping for the logical topology has been found; otherwise a survivable mapping does not exist.

Duality between circuits and cuts in a graph is one of the well studied topics in graph theory. This concept has played a significant role in the development of methodologies for solving problems in various applications. Most of the early results in electrical circuit theory were founded on the duality relationship between circuits and cuts [9]. There is a wealth of literature on the role of duality in network optimization (that is, discrete optimization on graphs and networks) [10]. Most often, for a primal algorithm based on circuits there is a dual algorithm based on cuts for the same problem. The primal and dual algorithms possess certain characteristics that make one superior to the other depending on the application. SMART algorithm for the survivable logical topology design problem is based on circuits [6, 7, 8].

The question then arises whether there exists a dual methodology based on cuts. The work in [11] answered this question in the affirmative and provided a unified algorithmic framework for the SLTM problem. The work also provides much insight into the structure of solutions for the SLTM problem.

Effectiveness of both these frameworks—SMART and its dual— as well as their robustness in providing survivability against multiple failures depends on the lengths of the cutset cover and circuit cover sequences on which they are based. In a recent unpublished work [12] we considered these issues and developed a generalized theory of logical topology survivability. We introduced the concept of generalized cutset and generalized circuit cover sequences. We showed that the distinction between the primal and dual methods disappears when the generalized sequences are used. Details of this work will be presented in the talk.

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Full orientability of graphs

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Graphs considered in this paper are finite and simple. For a graph $G = (V, E)$ with vertex set V and edge set E, we use $|V|$ and $|E|$ to denote the cardinalities of V and E, respectively. An orientation D of G is directed graph obtained by assigning a fixed direction on every edge xy of G. An orientation D is called *acyclic* if D does not contain any directed cycle.

Suppose that D is an acyclic orientation of G. An arc $u \to v$ of D, or its underlying edge, is called *dependent* (in D) if there exists a directed path in the directed graph $D' = D - (u \to v)$ from u to v. Define $d(D)$ as the number of dependent arcs in D. Let $d_{\min}(G)$ and $d_{\max}(G)$ be, respectively, the minimum and maximum value of $d(D)$ over all acyclic orientations D of G. It is known ([3]) that $d_{\max}(G) = |E| - |V| + k$ for a graph G having k components.

If any integer d satisfying $d_{\min}(G) \leq d \leq d_{\max}(G)$ is achievable as $d(D)$ for some acyclic orientation D of G, then G is said to be *fully orientable*. Otherwise, it is called *non-fully-orientable*. West [15] showed the following theorem.

Theorem 1 *Complete bipartite graphs are fully orientable.*

Let $\chi(G)$ denote the *chromatic number* of G and $g(G)$ denote the *girth* of G. And $g(G)$ is ∞ if G possesses no cycles. Fisher et al. [3] showed the following theorem.

Theorem 2 *If G is a connected graph with $\chi(G) < g(G)$, then G is fully orientable and $d_{\min}(G) = 0$.*

Lih, Lin, and Tong [7] gave the following theorem.

Theorem 3 *Every outerplanar graph G is fully orientable.*

A graph G is called *k-degenerate* if each subgraph H of G contains a vertex of degree at most k in H. Lai, Chang, and Lih [5] have established the fully orientability of 2-degenerate graphs.

Theorem 4 *If G is 2-degenerate, then G is fully orientable.*

Lai and Lih [6] gives further examples of fully orientable graphs, such as subdivisions of Halin graphs and graphs of maximum degree at most three. Collins and Tysdal [2] as well as Rödl and Thoma [14] also study some results about dependent arcs.

Let G be a connected graph. Use depth-first search to construct a spanning tree T of G and index the vertices as $v_1, v_2, \ldots, v_{|V(G)|}$ in the order of that search. Label each edge v_iv_j of $E(G) \setminus E(T)$ by the pair (i, j) with $i < j$. Then order these edges in lexicographical order by their labels. Observe there are $|E(G)| - |V(G)| + 1$ edges in $E(G) \setminus E(T)$. We also note that each edge in $E(G) \setminus E(T)$ joins a vertex with one of its ancestors on the tree T. Let k_T denote the least number t such that, when the first t edges of $E(G) \setminus E(T)$ in the above lexicographical order are removed, the remaining subgraph H of G is a graph with $d_{min}(H) = 0$. Since T is a graph with $d_{min}(T) = 0$, $k_T \le |E(G)| - |V(G)| + 1$. In [8], Lai, Lih, and Tong prove the following theorem.

Theorem 5 *If G is a connected graph and T is a spanning tree of G obtained by depth-first search, then every number d satisfying $k_T \le d \le |E(G)| - |V(G)| + 1$ is achievable as $d(D)$ for some acyclic orientation D of G.*

Corollary 6 *If G is a connected graph with $d_{\min}(G) \le 1$, then G is fully orientable.*

Let $K_{r(n)}$ denote the complete r-partite graph each of whose partite sets has n vertices. Chang, Lin, and Tong [1] determined the fully orientiability of $K_{r(n)}$ and proved that $K_{r(n)}$ is non-fully-orientable when $r \ge 3$ and $n \ge 2$. These are the only non-fully-orientable graphs known so far and they all have girth 3. An immediate consequence is that, when m is a composite number, there exist m-degenerate graphs which are non-fully-orientable. Moreover, it can be checked that $K_{3(2)}$ is the smallest non-fully-orientable graph. Any acyclic orientation of $K_{3(2)}$ has 4, 6, or 7 dependent arcs. Based upon the above data, we conclude this paper by posing the following open questions.

Question 1. For a given odd prime p, does there exist a non-fully-orientable p-degenerate graph that is not $(p-1)$-degenerate?

Question 2. For any given integer $g \ge 4$, does there exist a non-fully-orientable graph G whose girth is g?

Question 3. Does there exist a non-fully-orientable graph G whose $d_{\min}(G)$ is 2 or 3?

Question 4. $K_{3(2)}$ shows that a maximal planar graph can be non-fully-orientable. How to characterize all fully orientable planar graphs?

Question 5. How to characterize those complete multipartite graphs that are fully orientable?

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Clique irreducible and weakly clique irreducible graphs–A survey

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We consider only finite, simple graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$.

A clique of a graph G is a maximal complete subgraph of G. Some properties of cliques of a graph are discussed in [15]. A graph G is clique irreducible if every clique in G of size at least two, has an edge (called essential edge) which does not lie in any other clique of G and is clique reducible if it is not clique irreducible [13]. A graph G is clique vertex irreducible if every clique in G has a vertex which does not lie in any other clique of G and is clique vertex reducible if it is not clique vertex irreducible [1]. Every clique vertex irreducible graph is clique irreducible, but the converse need not be true. A clique which contains an essential edge is called an essential clique. A graph G is weakly clique irreducible [16] if every edge belongs to at least one essential clique and is weakly clique reducible, otherwise. The clique irreducible graphs form a subclass of weakly clique irreducible graphs.

In [13], it is proved that the interval graphs are clique irreducible. Wallis and Zhang [14] generalized this result and attempted to characterize clique irreducible graphs. A graph G is hereditary weakly clique irreducible if G and all its induced subgraphs are weakly clique irreducible. A forbidden subgraph characterization for G to be hereditary weakly clique irreducible is given in [16].

However, a characterization of weakly clique irreducible graphs is still open.

The join of two graphs G and H denoted by $G \vee H$ is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$. The distance between any two vertices u and v in a connected graph G, is denoted by $d_G(u, v)$ and the complement of a graph G, by G^c.

A graph G is complement reducible (cograph) if it can be reduced to edge less graphs by taking complements with in components [7]. Cographs can also be recursively defined as follows,

(1) K_1 is a cograph

(2) If G is a cograph, so is G^c and

(3) If G and H are cographs, so is $G \vee H$.

It is also known [12] that, G is a cograph if and only if G does not contain P_4-the path on four vertices, as an induced subgraph.

A graph G is distance hereditary, if $d_G(u, v) = d_H(u, v)$ for every connected induced subgraph H of G, where $u, v \in V(H)$ [9]. The distance hereditary graphs can also be obtained from K_1 by recursively,

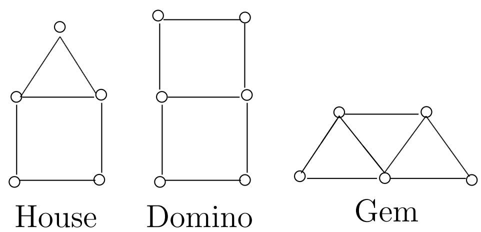
(1) Attaching pendant vertices

(2) Attaching true twins and

(3) Attaching false twins

where a true twin of a vertex u is a vertex v which is adjacent only to u and all its neighbors and a false twin of a vertex u' is a vertex v' which is adjacent only to the neighbors of u' [5].

The forbidden subgraphs of a distance hereditary graph are house, hole, domino and gem; where a hole is an odd cycle of length greater than or equal to five and other graphs are shown in the following figure [6].



It follows that, the cographs form a subclass of the distance hereditary graphs.

In [2], it is proved that if G^c has at least two non-trivial components then G is clique vertex reducible and if it has at least three non-trivial components then G is clique reducible. The cographs and the distance hereditary graphs which are clique vertex irreducible and clique irreducible are also recursively characterized. In [3], a recursive characterization and a forbidden subgraph characterization for the cographs and the distance hereditary graphs to be weakly clique irreducible are obtained.

The non-complete extended p-sum of graphs (NEPS) were first introduced in [8] in the context of studying eigen values of graphs. Let $\mathcal{B}$ be a non-empty subset of the collection of all binary n-tuples which does not include $(0, 0, ..., 0)$. The non-complete extended p-sum of graphs $G_1, G_2, ..., G_p$ with basis $\mathcal{B}$ denoted by NEPS$(G_1, G_2, ..., G_p; \mathcal{B})$, is the graph with vertex set $V(G_1) \times V(G_2) \times ... \times V(G_p)$, in which two vertices

$(u_1, u_2, ..., u_p)$ and $(v_1, v_2, ..., v_p)$ are adjacent if and only if there exists $(\beta_1, \beta_2, ..., \beta_p) \in \mathcal{B}$ such that u_i is adjacent to v_i in G_i whenever $\beta_i = 1$ and $u_i = v_i$ whenever $\beta_i = 0$. The graphs $G_1, G_2, ..., G_p$ are called the factors of the NEPS [8]. Most of the well known graph products are special cases of the NEPS.

There are seven possible ways of choosing the basis $\mathcal{B}$ when $p = 2$.

$\mathcal{B}_1 = \{(0,1)\}$
$\mathcal{B}_2 = \{(1,0)\}$
$\mathcal{B}_3 = \{(1,1)\}$
$\mathcal{B}_4 = \{(0,1),(1,0)\}$
$\mathcal{B}_5 = \{(0,1),(1,1)\}$
$\mathcal{B}_6 = \{(1,0),(1,1)\}$
$\mathcal{B}_7 = \{(0,1),(1,0),(1,1)\}$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs with $|V_i| = n_i$ and $|E_i| = m_i$ for $i = 1, 2$.

The NEPS$(G_1, G_2; \mathcal{B}_1)$ is G_2 repeated n_1 times and NEPS$(G_1, G_2; \mathcal{B}_2) =$ NEPS$(G_2, G_1; \mathcal{B}_1)$.

In the NEPS$(G_1, G_2; \mathcal{B}_j)$, two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if

(1) $j = 3$: u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2. This is same as the tensor product [10] of G_1 and G_2.

(2) $j = 4$: $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$. This is same as the cartesian product [10] of G_1 and G_2.

(3) $j = 5$: Either $u_1 = u_2$ or u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2.

(4) $j = 6$: This is same as NEPS$(G_2, G_1; \mathcal{B}_5)$.

(5) $j = 7$: Either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$ or u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2. This is same as the strong product [10] of G_1 and G_2.

In [2], the clique vertex irreducibility and clique irreducibility of graphs which are non-complete extended p-sums (NEPS) of two graphs are studied. Characterizations of G_1 and G_2 such that NEPS of G_1 and G_2 is clique irreducible and clique vertex irreducible are also obtained.

The line graph of a graph G is denoted by $L(G)$ and the iterations of $L(G)$ are recursively defined by $L^1(G) = L(G)$ and $L^{n+1}(G) = L(L^n(G))$, for $n \geq 1$ [14]. The

Gallai graph of a graph G, denoted by $\Gamma(G)$, is a graph whose vertex set corresponds to the edge set of G and any two vertices in $\Gamma(G)$ are adjacent if the corresponding edges in G are incident on a common vertex and they do not lie in a common triangle [11]. The anti-Gallai graph of a graph G, denoted by $\Delta(G)$, is a graph whose vertex set corresponds to the edge set of G and any two vertices in $\Delta(G)$ are adjacent if the corresponding edges lie in a triangle in G [11]. Both $\Gamma(G)$ and $\Delta(G)$ are spanning subgraphs of $L(G)$ and their union is $L(G)$. Though $L(G)$ has a forbidden subgraph characterization, both these graph classes cannot be characterized by forbidden subgraphs [11].

In [1], the graphs G for which $L(G)$ and $L^2(G)$ are clique vertex irreducible are characterized and it is deduced that $L^n(G)$ for $n \geq 3$ is clique vertex irreducible if and only if G is $K_3, K_{1,3}$ or P_k where $k \leq n+3$. We also prove that $L^n(G)$, $n \geq 5$, is clique irreducible if and only if it is non-empty and $L^4(G)$ is clique irreducible. The Gallai graphs which are clique irreducible and clique vertex irreducible are characterized. A forbidden subgraph characterization for clique vertex irreducibility of $\Gamma(G)$ is obtained. Also, the forbidden subgraphs for the anti-Gallai graphs and all its iterations to be clique irreducible and clique vertex irreducible are obtained.

In [17], the line graphs which are weakly clique irreducible are characterized. In [3], the Gallai graphs and the anti-Gallai graphs which are weakly clique irreducible are characterized. The anti-Gallai graphs posses an interesting property that, it is weakly clique irreducible only if it is clique irreducible. Also the closure property of weakly clique irreducible graphs with respect to NEPS of graphs are discussed in [4].

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Magic sum spectra of group magic graphs

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1 Background and introduction

1.1 Magic graphs

Various authors have introduced labelings that generalize the idea of the well-known magic square. In 1964, Sedláček [8] defined a graph to be **magic** if it had an edge labeling with range the real numbers, such that the sum of all incident edge labels around any vertex (vertex sum) equals some constant, independent of the choice of vertex. These labelings have been studied by Stewart in 1966-67, [12] and [13] for example, who called a labeling **supermagic** if the labels are consecutive integers starting from 1. Over several decades, variations of magic and anti-magic type of labeling have been studied, please see the up to date survey article [2] and two monographs [14] and [1] on these topics for more related details and applications.

1.2 Integer magic graphs

We call a graph to be A-**magic** if there exist edge labels ranging over the set A, such that the vertex sums are constant. It is known [4] that a graph G is $\mathbf{N}$-magic if and only if every edge of G is contained in a $\{1, 2\}$-factor and every pair of edges is separated by this $(1, 2)$-factor, and for a list of properties of $\mathbf{N}$-magic graphs, please see [3]. Richard Stanley studied $\mathbf{Z}$-magic graphs in [10, 11] regarding relations with commutative algebra, and he demonstrated that the study of magic labelings can be reduced to solving a system of linear diophantine equations. The case A being the finite group $\mathbf{Z_k}$ has been studied in literatures, for example [6, 9, 15], and sometimes referred as **integer magic** graphs. We consider in this article the $\mathbf{Z_k}$-magic graphs, and study the set of all possible magic vertex sum constants, namely the magic sum spectrum, for such graphs.

More precisely, for a positive integer $k \geq 2$, let $\mathbf{Z_k} = (\mathbf{Z_k}, +, \mathbf{0})$ be the additive group of integer congruences modulo k with identity 0. We call a finite simple graph G to be $\mathbf{Z_k}$-magic if it admits an edge labeling $f : E(G) \to \mathbf{Z_k}\backslash\{\mathbf{0}\}$ such that the induced vertex sum $f^+ : V(G) \to \mathbf{Z_k}$ defined by $f^+(v) = \sum_{uv\in E(G)} f(uv) = r$ is constant. The constant r is called a **magic sum index**, or an **index** for short, of G under the labeling f, which follows R. Stanley. For fix integer k, we denote by $I_k(G)$

the set of all magic sum indices r such that G is $\mathbf{Z_k}$-magic with an index r. We call $I_k(G)$ the **magic sum spectrum**, or the **index set**, of G with respect to $\mathbf{Z_k}$. Note that the case of $\mathbf{Z_2}$-magicness is easy to settle. It is not hard to see that a graph G is $\mathbf{Z_2}$-magic if and only if its degrees are of the same parity. However the discussion of $\mathbf{Z_2}$-magic graphs is completely different from that of $\mathbf{Z_k}$-magic graphs for $k \geq 3$. It is quite challenging, and still open thus far, to obtain nice characterizations for $\mathbf{Z_k}$-magic graphs for $k \geq 3$, see for example [6, 9, 15], and only certain necessary conditions for $\mathbf{Z_k}$-magicness are obtained. Therefore the index sets of $I_k(G)$ is harder to calculate for $k \geq 3$, since the information of index sets will imply that of $\mathbf{Z_k}$-magicness. Throughout this article we study the $\mathbf{Z_k}$-magicness for all $k \geq 3$, however, we remark that usually for the case of infinite cyclic group $\mathbf{Z}$ one may have similar results when one obtains results over the finite cyclic group $\mathbf{Z_k}$.

Note that, it is not hard to see that any regular graph is $\mathbf{Z_k}$-magic for $k \geq 3$, however it is not as easy to determine completely the magic sum spectrum for a regular graph G. We show in this paper that a regular graph with a 1-factor has the full index set $\mathbf{Z_k}$ for all $k \geq 3$, and give examples of regular graphs without 1-factor whose index set is not full $\mathbf{Z_k}$ for some $k \geq 3$. We also show that the index set of complete bipartite graphs $K_{m,n}$ is isomorphic to the cyclic subgroups $\mathbf{Z_d}$ generated by $\frac{k}{d}$, where $d = gcd(m-n, k)$. The index sets of Cartesian product and lexicographic product of graphs are studied. Among others we are able to determine completely the index sets of certain fundamental classes of graphs such as cycles C_n, complete graphs K_n, hypercubes Q_n, wheels W_n, fans F_n, complete bipartite graphs $K_{m,n}$, complete multipartite graphs $K_{n,n,\cdots,n}$ with parts of the same orders, and all circulant graphs.

2 Main results

Recall that $I_k(G)$ is the index set or the magic sum spectrum of G with respect to $\mathbf{Z_k}$ for $k \geq 3$. Among other results, we list the following:

Theorem 1 *Let G be an r-regular graph ($r \geq 2$) which admits a 1-factor, then $I_k(G) = \mathbf{Z_k}$ for all $k \geq 3$.*

Theorem 2

$$I_k(PC) = \begin{cases} \{0\}, & k = 3, \\ \mathbf{Z_4}\backslash\{0\}, & k = 4, \\ \mathbf{Z_k}, & k \geq 5. \end{cases}$$

where the graph PC please see the Figure 1.

Theorem 3 *Let C_n be an n-cycle, where $n \geq 3$, and k be a positive integer. We have the following:*

Figure 1: PC = Petersen's Example for Cubic Graphs without 1-Factor

1. $I_k(C_n) = 2(\mathbf{Z_k}\backslash\{0\}) = \{2x: \ x \neq 0, \ x \in \mathbf{Z_k}\}$, *for* n *odd.*

2. $I_k(C_n) = \mathbf{Z_k}$, *for* n *even.*

Theorem 4

$$I_k(CIR_n(S)) = \begin{cases} \mathbf{Z_k}, & \text{if there exists some } a \in S \text{ with } \frac{n}{gcd(a,n)} \text{ even.} \\ 2\mathbf{Z_k}, & \text{otherwise.} \end{cases}$$

where $CIR_n(S)$ *is the circulant graph with respect to* $S \subset \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, *which is defined as the graph with the vertex set* $V(CIR_n(S)) = \{0, 1, 2, \cdots, n-1\}$, *and the edge set is formed by the following rule:*

$$E(CIR_n(S)) = \{ij: \ i - j \equiv \pm s \pmod n, \ s \in S\}.$$

Theorem 5 *For* $m, n \geq 2$, *and* $k \geq 3$, *the index set* $I_k(K_{m,n})$ *of complete bipartite graph* $K_{m,n}$ *is* $\langle \frac{k}{d} \rangle$, *where* $d = gcd(m-n, k)$, *and* $\langle a \rangle$ *denote the additive subgroup of* Z_k *generated by the element* a. *Or equivalently,*

$$I_k(K_{m,n}) = \begin{cases} \{0\}, & \text{if } gcd(m-n,k) = 1. \\ \langle \frac{k}{d} \rangle \cong \mathbf{Z_d}, & \text{if } gcd(m-n,k) = d, \ 1 < d \leq k. \end{cases}$$

Theorem 6 *The index sets of fans* F_n, $n \geq 3$, *are as follows.*

when $k = 3$:

$$I_3(F_n) = \begin{cases} \emptyset, & n = 3. \\ \{0\}, & n = 4. \\ \emptyset, & n = 5. \\ \mathbf{Z_3}, & \text{for all } n \geq 6 \text{ and } n \equiv 1 \pmod 3. \\ \mathbf{Z_3}\backslash\{0\}, & \text{for all } n \geq 6 \text{ and } n \equiv 0, 2 \pmod 3. \end{cases}$$

when $k = 4$:

$$I_4(F_n) = \begin{cases} \{0, 2\}, & n = 3. \\ 2\mathbf{Z_4} = \{0, 2\}, & n \text{ even.} \\ \mathbf{Z_4}, & n \geq 5 \text{ odd.} \end{cases}$$

when $k \geq 5$:

$$I_k(F_n) = \begin{cases} \mathbf{Z_k}, & \textit{for all } n \geq 5 \textit{ odd , and } k \geq 5. \\ 2\mathbf{Z_k}, & \textit{for all } n \geq 4 \textit{ even , and } k \geq 5. \end{cases}$$

Theorem 7 *The index sets of wheels* W_n, $n \geq 3$, *are as follows.*

when $k = 3$:

$$I_3(W_n) = \begin{cases} \mathbf{Z_3}, & n \equiv 0 \ (mod\ 3). \\ \mathbf{Z_3} \backslash \{0\}, & n \equiv 1, 2 \ (mod\ 3). \end{cases}$$

when $k \geq 4$:

$$I_k(W_n) = \begin{cases} \mathbf{Z_k}, & \textit{for all } n \textit{ even .} \\ 2\mathbf{Z_k}, & \textit{for all } n \textit{ odd .} \end{cases}$$

Remark. Note that $2\mathbf{Z_k} = \mathbf{Z_k}$ when k is odd, and $2\mathbf{Z_k} = \{0, 2, \cdots, \frac{k}{2}\}$ when k is even.

3 Open problems

We list here the following open problems to be explored further inspired from our work:

1. Is the sufficient condition admitting a 1-factor is also necessary for regular graphs to have the full index sets $\mathbf{Z_k}$, for all $k \geq 3$? See the examples of the index sets of regular graphs without 1-factor in this article, which provide positive evidence for the above characterization that the sufficient condition is also the necessary condition.

2. When is the index set $I_k(G)$ a subgroup of $\mathbf{Z_k}$?

3. Characterize the graphs G having the index sets $I_k(G) = \mathbf{Z_d}$, where d is a divisor of k. Note that this open problem includes cases $d = 1$, $1 < d < k$, and $d = k$, respectively. The case $d = 1$ corresponds to characterizing the graphs G having the index sets $I_k(G) = \{0\}$, for all $k \geq 3$. The case $d = k$ corresponds to characterizing the graphs G having the index sets $I_k(G) = \mathbf{Z_k}$, for all $k \geq 3$.

4. Characterize the graphs which is $\mathbf{Z_k}$-magic, for $k \geq 3$.

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D-bounded distance-regular graphs

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Let Γ=(X, R) denote a finite undirected, connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter D:=max$\{\,\partial(x,y)\mid x,y\in X\}$. For a vertex $x\in X$ and an integer $0\le i\le D$, set $\Gamma_i(x):=\{\,z\in X\mid \partial(x,z)=i\}$. The *valency* $k(x)$ of a vertex $x\in X$ is the cardinality of $\Gamma_1(x)$. The graph Γ is called *regular* (with *valency* k) if each vertex in X has valency k. A graph Γ is said to be *distance-regular* whenever for all integers $0\le h,i,j\le D$, and all vertices $x,y\in X$ with $\partial(x,y)=h$, the number

$$p^h_{ij}=|\Gamma_i(x)\cap\Gamma_j(y)|$$

is independent of x,y. The constants p^h_{ij} are known as the *intersection numbers* of Γ.

From now on let $\Gamma=(X,R)$ be a distance-regular graph with diameter $D\ge 3$. For two vertices $x,y\in X$, with $\partial(x,y)=i$, set

$$\begin{aligned}B(x,y)&:=\Gamma_1(x)\cap\Gamma_{i+1}(y),\\ C(x,y)&:=\Gamma_1(x)\cap\Gamma_{i-1}(y),\\ A(x,y)&:=\Gamma_1(x)\cap\Gamma_{i}(y).\end{aligned}$$

Note that

$$\begin{aligned}|B(x,y)|&=p^i_{1\ i+1},\\ |C(x,y)|&=p^i_{1\ i-1},\\ |A(x,y)|&=p^i_{1\ i}\end{aligned}$$

are independent of x, y. For convenience, set $c_i:=p^i_{1\ i-1}$ for $1\le i\le D$, $a_i:=p^i_{1\ i}$ for $0\le i\le D$, $b_i:=p^i_{1\ i+1}$ for $0\le i\le D-1$ and put $b_D:=0$, $c_0:=0$, $k:=b_0$. Note that k is the valency of Γ.

Recall that a sequence x, z, y of vertices of Γ is *geodetic* whenever

$$\partial(x,z)+\partial(z,y)=\partial(x,y),$$

where ∂ is the distance function of Γ. A sequence x, z, y of vertices of Γ is *weak-geodetic* whenever

$$\partial(x,z)+\partial(z,y)\le\partial(x,y)+1.$$

Definition 1. A subset $\Delta\subseteq X$ is *weak-geodetically closed* if for any weak-geodetic sequence x, z, y of Γ,

$$x,\ y\in\Delta\Longrightarrow z\in\Delta.$$

Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [6]. If a weak-geodetically closed subgraph Δ of diameter d is regular then it has valency $a_d+c_d=b_0-b_d$, where a_d, c_d, b_0, b_d are intersection numbers of Γ. Furthermore Δ is distance-regular with intersection numbers $a_i(\Delta)=a_i(\Gamma)$ and $c_i(\Delta)=c_i(\Gamma)$ for $1\le i\le d$ [10, Theorem 4.5].

Definition 2. Γ is said to be *i-bounded* whenever for all $x,y\in X$ with $\partial(x,y)\le i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x,y)$ which contains x and y.

Note that a $(D-1)$-bounded distance-regular graph is clear to be D-bounded. The properties of D-bounded distance-regular graphs were studied in [11], and these properties were used in the classification of classical distance-regular graphs of negative type [12]. Before stating our main result we make one more definition.

By a *parallelogram of length i*, we mean a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x,y)=\partial(z,w)=1$, $\partial(x,w)=i$, and $\partial(x,z)=\partial(y,w)=\partial(y,z)=i-1$. The previous study of parallelogram-free distance-regular graphs can be found in [3, 7, 9]. The following theorem is our main result in this talk.

Theorem 3 *Let Γ denote a distance-regular graph with diameter $D\ge 3$, and intersection numbers $a_1=0$, $a_2\ne 0$. Fix an integer $1\le d\le D-1$ and suppose Γ contains no parallelograms of any length up to $d+1$. Then Γ is d-bounded.*

Theorem 3 is a generalization of [1, Lemma 4.3.13], [4], and is also proved under an additional assumption $c_2>1$ by A. Hiraki [2]. To prove Theorem 3, we need many previous results of [2]. Theorem 3 also answers the problem proposed in [10, p. 299]. Many previous results prove its complement case $a_1\ne 0$, for examples under an additional assumption $c_2>1$ [10] and under the assumptions $a_2>a_1>c_2=1$

[8]. For the assumptions $a_2 > a_1$ and $c_2 = 1$, H. Suzuki proves the case $d = 2$ in Theorem 3 [8]; in particular Γ contains a regular weak-geodetically closed subgraph Ω of diameter 2. Since the Friendship Theorem [13, Theorem 8.6.39] asserts no such Ω in the case $a_1 = c_2 = 1$, there must be no such distance-regular graph Γ with $a_2 > a_1 = c_2 = 1$ and Γ contains no parallelograms of length 3. Note that the assumption $a_1 \neq 0$ implies $a_2 \neq 0$ [1, Proposition 5.5.1(i)]. Hence Theorem 3 is also true under the weaker assumptions $b_1 > b_2$ and $a_2 \neq 0$. Our method in proving Theorem 3 also works for the case $b_1 > b_2$ and $a_2 \neq 0$ after a slight modification, but we decide not to duplicate the previous works.

On the other hand we suppose that Γ is d-bounded for $d \geq 2$. Let $\Omega \subseteq \Delta$ be two regular weak-geodetically closed subgraphs of diameters $1, 2$ respectively. Since Ω and Δ have different valency $b_0 - b_1$ and $b_0 - b_2$ respectively, we have $b_1 > b_2$. It is also easy to see that Γ contains no parallelograms of any length up to $d + 1$ [10, Lemma 6.5]. With these comments, Theorem 3 is the final step in the following characterization of d-bounded distance-regular graphs in terms of forbidden parallelograms.

Theorem 4. *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Suppose the intersection number $a_2 \neq 0$. Fix an integer $2 \leq d \leq D - 1$. Then the following two conditions (i), (ii) are equivalent:*

(i) Γ is d-bounded.

(ii) Γ contains no parallelograms of any length up to $d + 1$ and $b_1 > b_2$.

Some applications of Theorem 3 were previously given in [2], [5].

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A study on bandwidth sum of join of two graphs

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Given a graph G, a proper labeling f of G is a one-to-one function $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$. The bandwidth sum (resp. bandwidth) of a graph G with respect to f is defined by $\mathrm{BS}_f(G) = \sum_{uv\in E(G)} |f(u)-f(v)|$ (resp. $B_f(G) = \max_{uv\in E(G)} |f(u)-f(v)|$). The bandwidth sum (resp. bandwidth) of a graph G, is defined by $\mathrm{BS}(G) = \min \mathrm{BS}_f(G)$ (resp. $B(G) = \min B_f(G)$), where the minimum is taken for all proper labelings f of G. For a proper labeling f, the profile width $w_f(v)$ of a vertex v in a graph G is

$$w_f(v) = \max_{x\in N[v]} (f(v) - f(x)).$$

The profile $P_f(G)$ of f is defined by

$$P_f(G) = \sum_{v\in V(G)} w_f(v),$$

and the profile of G is $P(G) = \min\{P_f(G)|f \text{ is a proper labeling of } G\}$. The join of G and H is the graph $G + H$ with $(G + H) = V(G) \cup V(H)$ and

$$E(G + H) = E(G) \cup E(H) \cup \{uv|\ u \in V(G) \text{ and } v \in V(H)\}.$$

Garey et al. [5] showed that the bandwidth problem is NP-complete for general graph. In fact, they proved that the problem is also NP-complete even when G is restricted to the class of trees with no degree exceeding three. Liu, Wang, and Williams [12] gives bounds for $B(G+H)$. Lai, Liu, and Williams [11] also provides the bounds of bandwidth for the join of k Graphs. Yan [15] find $B(G+H)$ and it gives a linear-time algorithm for finding the bandwidth for cographs. The profile problem is NP-complete since it is equivalent to the interval graph completion problem [13]. Kuo and Chang [9] gave a polynomial-time algorithm for finding the profile problem of a tree. Kuo [8] find $P(G + H)$ and it gives a linear-time algorithm for finding profile of a cograph.

For bandwidth sum problem, Garey and Johnson [6] proved that the problem is NP-complete for general graphs. Chung [4] gave a polynomial-time algorithm for finding the bandwidth sum of a tree. What is $\mathrm{BS}(G+H)$? Lai and Williams [10] gave an algorithm for solving $\mathrm{BS}(G_1+G_2+...+G_k)$ when each G_i is "sum deterministic". Chen, Kuo, Lin, and Yan [2] find $\mathrm{BS}(G_1 + G_2 + ... + G_k)$ when each G_i is a path, cycle, complete graph, or union of isolated vertices. Chia, Kuo, and Yan [3] find $\mathrm{BS}(K_1 + (\bigcup_{i=1}^{r} K_{n_i}))$. Can we find a polynomial-time algorithm for solving $\mathrm{BS}(G + H)$

when G and H are general graphs and finding bandwidth sum of a cograph. We have some results for $\mathrm{BS}(K_1 + G)$.

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A study of the edge span of distance three labeling

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1 Introduction

The distance labeling (or coloring) of a graph is an assignment of numbers (or labels) to the vertices with conditions depend on the distance between vertices. This class of graph labeling is motivated by the frequency assignment problem.There are considerable efforts on this labeling since it was introduced in 1992.

Given a graph $G = (V, E)$ and nonnegative integers $k_1 \geq k_2 \geq k_3$, an $L(k_1, k_2, k_3)$-*labeling* of G is an assignment $f : V \to \{0, 1, \dots\}$ such that $|f(u) - f(v)| \geq k_i$ whenever the distance between u and v is i in G, for $i = 1, 2, 3$. The tuple (k_1, k_2, k_3) is called the *constraint* of the labeling. The $L(k_1, k_2, k_3)$-*span*, $\lambda(G; k_1, k_2, k_3)$, is the smallest number m such that there is an $L(k_1, k_2, k_3)$-labeling with the maximum value m. If $k_3 = 0$ then the constraint is denoted by (k_1, k_2) for short. In the previous study, people are interested in finding $L(k_1, k_2)$-spans with various k_1 and k_2. Analogously, we consider another parameter of $L(k_1, k_2, k_3)$-labeling. Given an $L(k_1, k_2, k_3)$-labeling f of a graph G, the *edge span* of f is defined by $\max\{|f(u) - f(v)| : uv \in E(G)\}$. The $L(k_1, k_2, k_3)$-*edge span* of G is the minimum edge span over all $L(k_1, k_2, k_3)$-labelings of G and is denoted by $\beta(G; k_1, k_2, k_3)$. The case with constraint $(2, 1, 0)$ has been studied by Yeh[1].

In a communication network, large service areas are often covered by a network of congruent polygonal cells with each station or transmitter at the center of cell that it covers. There are only three regular tilings (regular cell coverings) can cover the whole plane, which are square tiling, hexagonal tiling and triangular tiling. Correspondingly, we have the square lattice, the triangular lattice and the hexagonal lattice.

Since our labeling problem was motivated by the channel assignment problem of a communication network,this talk will present recent results on edge spans with constraints (k_1, k_2, k_3) for these three classes of graphs.

2 Properties and Basic Results

The following properties of the edge span β are easy to obtain.

1. If H is a subgraph of G then $\beta(H; k_1, k_2, k_3) \le \beta(G; k_1, k_2, k_3)$
2. If $d_i \ge k_1$ for $i = 1, 2, 3$ then $\beta(G; d_1, d_2, d_3) \ge \beta(G; k_1, k_2, k_3)$.
3. $k_1 \le \beta(G; k_1, k_2, k_3) \le \lambda(G; k_1, k_2, k_3)$.
4. If G is a r-regular graph then $\beta(G; k_1, k_2, k_3) \ge k_1 + (r-1)k_2$.

Let P_n and C_n be path and cycle of order n. Then

1. $\beta(K_n; k_1, k_2, k_3) = (n-1)k_1 = \lambda(K_n; k_1, k_2, k_3)$.
2. $\beta(P_n; k_1, k_2, k_3) = k_1$, where $n \ge 2$.
3. $\beta(C_n; k_1, k_2, 1) = k_1 + k_2$ if $k_2 \le k_1 \le 3k_2$, where $n \ge 4$.

3 Infinite Graphs

Let Γ_Z, Γ_S, Γ_T and Γ_H be two-side infinite path, square lattice, triangular lattice and hexagonal lattice, respectively. Then we have the following results:

1.
$$\beta(\Gamma_Z; k_1, k_2, k_3) = k_1 + k_2.$$

2.
$$\beta(\Gamma_S; k_1, k_2, 1) = \begin{cases} 5 & \text{if } k_1 = k_2 = 1, \\ k_1 + 3k_2 & \text{if } k_1 \ne k_2, 2k_2, 3k_2. \end{cases}$$

3.
$$\beta(\Gamma_T; 3, 2, 1) = 16.$$

4.
$$\beta(\Gamma_H; k_1, k_2, 1) = \begin{cases} 4 & \text{if } k_1 = k_2 = 1, \\ k_1 + 2k_2 & \text{otherwise.} \end{cases}$$

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Total weight choosability of graphs

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Suppose $G = (V, E)$ is a graph. For a vertex x of G, $E(x)$ denotes the set of edges of G incident to x. A (proper) *edge weighting* of G is a mapping that assigns to each edge e of G a real number $f(e)$ so that for any edge xx' of G, $\sum_{e \in E(x)} f(e) \neq \sum_{e \in E(x')} f(e)$. The real number assigned to an edge is called the *weight* of the edge. Instead of real numbers, one can use elements of some other field as weights. For simplicity, the weights are restricted to real numbers in this paper. The study of edge weighting is initiated by Karoński, Łuczak and Thomason [11]. They proposed the following conjecture in [11]:

Conjecture 1 Every connected graphs $G \neq K_2$ has an edge weighting f such that $f(e) \in \{1, 2, 3\}$ for every edge e.

This conjecture is referred as the $1, 2, 3$-conjecture and has received some attention. It is shown in [11] if S is a subset of R of size at least 183 and is independent over the field of rational numbers, then any connected graph $G \neq K_2$ has an edge weighting f with $f(e) \in S$ for every edge e. Under the same restriction that S being independent over rational numbers, the bound 183 is reduced to 4 in [1] (if the minimum degree is at least 1000 then the bound is reduced to 3). In [3], it is shown that any connected graph $G \neq K_2$ has an edge weighting f with $f(e) \in \{1, 2, \ldots, 30\}$ for every edge e. The result was improved in [2], where it is shown that f can be chosen so that $f(e) \in \{1, 2, \ldots, 16\}$ for every edge e.

In [7], Bartnicki, Grytczuk and Niwczyk considered the choosability version of $1, 2, 3$-conjecture. A graph is said to be *k-edge-weight-choosable* if the following is true: For any list assignment L which assigns to each edge e a set $L(e)$ of k real numbers, G has an edge weighting f such that $f(e) \in L(e)$ for each edge e.

Conjecture 2 Every graph without isolated edges is 3-edge-weight-choosable.

Conjecture 2 is stronger than Conjecture 1. Bartnicki, Grytczuk and Niwczyk [7] verified this conjecture for complete graphs, complete bipartite graphs and some other graphs. However, it is unknown if there is a constant k such that every graph without isolated edges is k-edge-weight-choosable.

Przybyło and Woźniak [12, 13] studied weighting that involves both the edges and the vertices of G. Suppose $G = (V, E)$ is a graph. A mapping $f : V \cup E \to R$ is called a (proper) *total weighting* of G if the *vertex-colouring* ϕ_f *of* G *induced by* f defined as

$$\phi_f(x) = \sum_{e \in E(x)} f(e) + f(x)$$

is a proper colouring of G, i.e., for any two adjacent vertices x and x', $\phi_f(x) \neq \phi_f(x')$. Przybyło and Woźniak proposed the following conjecture and named it the $1, 2$-conjecture in [12]:

Conjecture 3 Every simple graph G has a total weighting f such that $f(y) \in \{1, 2\}$ for all $y \in V \cup E$

Przybyło and Woźniak [12, 13] verified this conjecture for some special graphs, including complete graphs, 4-regular graphs and graphs G with $\chi(G) \leq 3$. They also proved that every simple graph G has a total weighting f such that $f(y) \in \{1, 2, \ldots, 11\}$ for all $y \in V \cup E$.

In this talk, we consider the choosability version of total weighting. A *total list assignment* of G is a mapping $L : V \cup E \to \mathcal{P}(R)$ which assigns to each element $y \in V \cup E$ a set $L(y)$ of real numbers as *permissible weights*. Given a total list assignment L, a total weighting f is called an L*-total weighting* if for each $y \in V \cup E$, $f(y) \in L(y)$. We say G is L*-total weightable* if there exists a L-total weighting f of G. Given a pair (k, k') of positive integers, a total list assignment L is called a (k, k')-total list assignment if $|L(x)| = k$ for each vertex $x \in V$ and $|L(e)| = k'$ for each edge $e \in E$. We say G is (k, k')*-total weight choosable* ((k, k')-choosable, for short) if for any (k, k')-total list assignment L, G is L-total-weightable.

The notion of L-total weightability and the notion of (k, k')-total weight choosable is a common generalization of the above mentioned various edge weighting concepts. It also includes the ordinary vertex colouring and vertex choosability as its special cases. A total list assignment L is called a *vertex list assignment* if $L(e) = \{0\}$ for each edge $e \in E$. If L is a vertex list assignment for which $|L(x)| = k$ for each vertex x, then L is a vertex k-list assignment. A graph G is called k*-choosable* if for any k-vertex list assignment L, G is L-total weightable. The *choosability* $ch(G)$ of a graph G (also called the choice number of G or the list chromatic number of G and denoted by $\chi_l(G)$) is the minimum integer k such that G is k-choosable.

It follows from the definition that if a graph G is $(k, 1)$-choosable, then G is k-choosable (as one can let $L(e) = \{0\}$ for each edge e). The converse is also true. Assume G is k-choosable, and L is a $(k, 1)$-total list assignment with $L(e) = \{l(e)\}$ for each edge e. Let L' be a k-vertex list assignment defined as $L'(x) = \{a + \sum_{e \in E(x)} l(e) : a \in L(x)\}$. Since G is k-choosable, G has an L'-colouring f'. Then $f(x) = f'(x) - \sum_{e \in E(x)} l(e)$ and $f(e) = l(e)$ is an L-total weighting of G.

It also follows from the definition that if a graph G is $(1, k)$-choosable, then it is k-edge-weight-choosable. The converse is not true. For example, the path $P = (v_1, v_2, v_3, v_4)$ is 2-edge-weight-choosable, but it is not $(1, 2)$-choosable. Let $L(v_1) = \{1\}$, $L(v_i) = \{0\}$ for $i = 2, 3, 4$, and let $L(e) = \{0, 1\}$ for each edge e. It can be verified that there is no L-total weighting. However, we shall show that those graphs shown in [7] to be k-edge-weight-choosable are also $(1, k)$-choosable. We propose two conjectures concerning total weight choosability of graphs.

Conjecture 4 There are constants k, k' such that every graph is (k, k')-choosable.

Conjecture 5 Every graph is $(2, 2)$-choosable. Every graph with no isolated edges is $(1, 3)$-choosable.

Note that if a graph G is (k, k')-choosable, then it is $(k + 1, k')$-choosable and $(k, k' + 1)$-choosable. If Conjecture 5 is true, then the integer pairs are best possible, as there are graphs which are not $(1, 2)$-choosable and for any integer k, there are graphs which are not $(k, 1)$-choosable.

Conjecture 5 is stronger than Conjectures 1, 2 and 3. Although $(1, 3)$-total weight choosability is stronger than the 3-edge weight choosability, we shall see that in some inductive proofs, the stronger version has its advantage. The argument in [7] actually shows that complete graphs, complete bipartite graphs and some other graphs are $(1, 3)$-choosable. In this talk, I shall survey some the results concerning total choosability of graphs obtained in a few papers (by Bartnicki, Grytczuk and Niwczyk, Wong, Yang and Zhu, Chang, Wong and Zhu). Then I shall explain how to prove that the complete graphs are $(2, 2)$-choosable, by using combinatorial nullstellensatz.

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[15] T. Wong, D. Yang and X. Zhu, *Total weighting of graph with max-min method*, preprint, 2008.

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You can make Professors Arumugam, Thulasiraman, Sen, Subir Ghosh, Ravindra and Paulraja as Session Chairs.

November 9 (Monday)

08:30–09:00	Registration
	Session Chair: Gerard Jennhwa Chang (National Taiwan University)
09:00–09:40	S. Arumugam (National Center for Advanced Research in Discrete Mathematics) Component-complete sets in graphs (page 1)
09:40–10:20	Sheng-Chyang Liaw (廖勝强) (National Central University) Global defensive alliances in double-loop networks (page 70)
10:20–10:50	Tea Time
	Session Chair: S. Arumugam (National Center for Advanced Research in Discrete Mathematics)
10:50–11:30	T. Tamizh Chelvam (Manonmaniam Sundaranar University) Total and connected domination in circulant graphs (page 22)
11:30–12:10	Chiang Lin (林强) (National Central University) Minimum statuses of connected graphs (page 75)
12:10–13:30	Lunch Time
	Session Chair: Ko-Wei Lih (李國偉) (Academia Sinica)
13:30–14:10	R. Balakrishnan (Bharathidasan University) b-coloring of Kneser graphs (page 12)
14:10–14:50	S. Francis Raj (Bharathidasan University) b-coloring of graphs (page 93)
14:50–15:30	Kuo-Ching Huang (黃國卿) (Aletheia University) On the equitable colorings of Kneser graphs (page 55)
15:30–16:00	Tea Time
	Session Chair: R. Balakrishnan (Bharathidasan University)
16:00–16:40	Ko-Wei Lih (李國偉) (Academia Sinica) Adjacent vertex distinguishing edge-colorings of planar graphs with girth at least six (page 72)
16:40–17:20	Roger K. Yeh (葉光清) (Feng Chia University) A study of the edge span of distance three labeling (page 131)
17:00–18:00	Xuding Zhu (朱緒鼎) (National Sun Yat-sen University) Total weight choosability of graphs (page 133)

November 10 (Tuesday)

08:30–09:00	Registration
	Session Chair: Xuding Zhu (朱緒鼎) (National Sun Yat-sen University)
09:00–09:40	Krishnaiyan Thulasiraman (University of Oklahoma) Duality in graphs and logical topology survivability in layered networks (page 107)
09:40–10:20	Hung-Lin Fu (傅恆霖) (National Chiao Tung University) On optimal pebbling of hypercubes (page 31)
10:20–10:50	Tea Time
	Session Chair: Krishnaiyan Thulasiraman (University of Oklahoma)
10:50–11:30	Tayuan Huang (黃大原) (National Chiao Tung University) An extension of Stein-Lovász Theorem and some of its applications (page 58)
11:30–12:10	Sen-Peng Eu (游森棚) (National University of Kaohsiung) Cyclic sieving for cyclic polytopes (page 28)
12:10–13:30	Lunch Time
	Session Chair: Hung-Lin Fu (傅恆霖) (National Chiao Tung University)
13:30–14:10	G. Ravindra (NCERT) Strongly and co-strongly perfect graphs (page 95)
14:10–14:50	A. Vijayakumar (Cochin University of Science and Technology) Clique irreducible and weakly clique irreducible graphs–A survey (page 114)
14:50–15:30	Chih-wen Weng (翁志文) (National Chiao Tung University) D-bounded distance-regular graphs (page 124)
15:30–16:00	Tea Time
	Session Chair: G. Ravindra (NCERT)
16:00–16:40	T. Karthick (Indian Institute of Technology Madras) On $\{P_2 \cup P_3, C_4\}$-free graphs (page 62)
16:40–17:20	Manoj Changat (University of Kerala) On a new class of graphs from posets: The cover-incomparability graphs (page 19)

November 11 (Wednesday)

08:30–09:00	Registration
	Session Chair: Tayuan Huang (黃大原) (National Chiao Tung University)
09:00–09:40	Subir Kumar Ghosh (Tata Institute of Fundamental Research) On joint triangulations of two sets of points in the plane (page 34)
09:40–10:20	Partha Pratim Goswami (Tata Institute of Fundamental Research) Unsolved problems in visibility graph theory (page 44)
10:20–10:50	Tea Time
	Session Chair: Subir Kumar Ghosh (Tata Institute of Fundamental Research)
10:50–11:30	Subhas C. Nandy (Indian Statistical Institute) Algorithmic issues in the intersection graphs of 2D objects and their applications (page 80)
11:30–12:10	Hsun-Wen Chang (張薰文) (Tatung University) Joint structural importance in consecutive-k-out-of-n systems (page 15)
12:10–13:30	Lunch Time

November 12 (Thursday)

08:30–09:00	Registration
	Session Chair: Chiuyuan Chen (陳秋媛) (National Chiao Tung University)
09:00–09:40	Sandeep Sen (Indian Institute of Technology Delhi) A simple linear time algorithm for constructing spanners in weighted graphs (page 99)
09:40–10:20	Ramesh Krishnamurti (Simon Fraser University) A primal-dual algorithm for the unconstrained fractional matching problem (page 65)
10:20–10:50	Tea Time
	Session Chair: Sandeep Sen (Indian Institute of Technology Delhi)
10:50–11:30	Tao-Ming Wang (王道明) (Tunghai University) Magic sum spectra of group magic graphs (page 119)
11:30–12:10	Li-Da Tong (董立大) (National Sun Yet-sen University) Full orientability of graphs (page 111)
12:10–13:30	Lunch Time
	Session Chair: Tao-Ming Wang (王道明) (Tunghai University)
13:30–14:10	P. Paulraja (Annamalai University) C_k-Decompositions of complete equipartite graphs (page 90)
14:10–14:50	A. Muthusamy (Kongu Engineering College) $\bar{S}_k$-factorization of symmetric digraph of product graphs (page 77)
14:50–15:30	Chiuyuan Chen (陳秋媛) (National Chiao Tung University) Constructing independent spanning trees for hypercubes and locally twisted cubes (page 25)
15:30–16:00	Tea Time
	Session Chair: P. Paulraja (Annamalai University)
16:00–16:40	R. Sampathkumar (Annamalai University) Orthogonal double covers of graphs and mutually orthogonal graph squares (page 96)
16:40–17:20	Jing-Ho Yan (顏經和) (Aletheia University) A study on bandwidth sum of join of two graphs (page 128)