Note

Maximal sets of Hamilton cycles in $D_n$

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Abstract

In this paper, we prove that there exists a maximal set of $m$ directed Hamilton cycles in $D_n$ if and only if $\left\lceil \frac{n}{2} \right\rceil \leq m \leq n - 1$ for $n \geq 7$.

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1. Introduction

A Hamilton cycle in a graph $T$ is a spanning cycle of $T$. If $S$ is a set of edge-disjoint Hamilton cycles in $T$ and if $E(S)$ is the set of edges occurring in the Hamilton cycles in $S$, then $S$ is said to be maximal if $T - E(S)$ has no Hamilton cycle.

In paper [5], Hoffman et al. showed that there exists a maximal set $S$ of $m$ edge-disjoint Hamilton cycles in the complete graph $K_n$ if and only if $\lceil (n+3)/4 \rceil \leq m \leq \lfloor (n-1)/2 \rfloor$. In paper [1], Bryant et al. showed that there exists a maximal set $S$ of $m$ edge-disjoint Hamilton cycles in the complete bipartite graph $K_{n,n}$ if and only if $n/4 < m \leq n/2$. Later, Daven et al. [2] showed for $n \geq 3$ and $p \geq 3$, there exists a maximal set $S$ of $m$ Hamilton cycles in the complete multipartite graph $K_{n}^{p}$ ($p$ parts of size $n$) if and only if $\lfloor (p-1)/2 \rfloor \leq m \leq \lfloor (n+1)(p-1)-2 \rfloor/4$, and $m > n(p-1)/4$ if $n$ is odd and $p \equiv 1 \pmod{4}$, except possibly if $n$ is odd and $m \leq (n+1)(p-1)-2)/4$. Recently, Fu et al. [3] extended the results in [2] and proved that if $\lceil (p-1)/2 \rceil \leq m \leq p-1$, then there exists a maximal set $S$ of $m$ Hamilton cycles in $K_{2p} - F$, where $F$ is a 1-factor of $K_{2p}$.

In this paper, we will extend these results to directed graphs and study maximal sets of Hamilton cycles in the complete directed graph $D_n$.

Throughout this paper, we will use the following notation and terminology. A digraph $D$ is said to be $r$-regular if $\deg^+(v) = \deg^-(v) = r$ for each vertex $v$ in $D$. Let $D_n$ be a complete directed graph of order $n$ (without loops); it is clear that $D_n$ is $(n-1)$-regular. For disjoint sets $A$ and $B$, define the directed complete bipartite graph $D_{A,B}$ as the
2. Maximal sets of Hamilton cycles in $D_n$

Before proving the main result, we need the following lemma.

Lemma 2.1. For an integer $n$, $n \geq 7$, $D_n$ has a Hamilton decomposition.

Proof. If $n$ is odd, $D_n$ has a Hamilton decomposition [7]. When $n$ is even, Tillson [8] gives the proof. □

The following lemma plays an important part to prove our main theorem.

Lemma 2.2. Let $D$ be an $r$-regular directed graph on $n$ vertices. If $r \geq [n/2]$, then $D$ contains a directed Hamilton cycle.

In fact, this result is a special case of the following theorem which is obtained by Ghouila-Houri [4].

Theorem 2.1 (Ghouila-Houri [4]). Let $\delta^+(D)$ and $\delta^-(D)$ denote the minimum outdegree and minimum indegree in $D$, respectively. If $D$ is a strict digraph (without cycles of length 2), and $\min\{\delta^+(D), \delta^-(D)\} \geq |V(D)|/2$, then $D$ is Hamiltonian.

On the other hand, if $D$ is regular strict digraph, then the condition for $D$ to be Hamiltonian can be weaken slightly. The following result was obtained by Jackson.

Lemma 2.3 (Jackson [6]). Every strict digraph of minimum in-degree and out-degree $k \geq 2$, on at most $2k + 2$ vertices, is Hamiltonian.

Now, we are ready to prove our main result. In order to construct maximal sets of Hamilton cycles. By Lemma 2.3, we conclude that the minimum number $m$ in a maximal set of $m$ directed Hamilton cycles in $D_n$ is $[n/2]$. Therefore, it suffices to construct a maximal set of $m$ directed Hamilton cycles for each $[n/2] \leq m \leq n - 1$ for $n \geq 7$. In order to do that, the following lemma is essential. Mainly, we make an arrangement of directed differences in order that we can construct maximal sets of Hamilton cycles systematically.

Lemma 2.4. For positive integers $l$ and $m$ where $m \geq 5$ and $1 \leq l \leq m - 1$, there exists an ordered set $A_{m-l} = (a_1, a_2, \ldots, a_{m-l}) \subseteq \mathbb{Z}_m$ such that $S = \{(a_{t+l} - a_t) \mod m | t = 1, 2, \ldots, m - l - 1\}$ is an $(m - l - 1)$-subset of $\{1, 2, \ldots, m - 1\}$ (all directed differences are distinct).

Proof. We divide the proof into two cases.

Case 1: When $m$ is even.

When $l = 1$, we delete the last element from the following ordered set $(0, 1, m - 1, 2, m - 2, 3, \ldots, m/2 - 2, m/2 + 2, m/2 - 1, m/2 + 1, m/2)$ to get $A_{m-1}$, i.e.,

$$A_{m-1} = \left(0, 1, m - 1, 2, m - 2, 3, m - 3, \ldots, \frac{m}{2} - 2, \frac{m}{2} + 2, \frac{m}{2} - 1, \frac{m}{2} + 1\right).$$
When \( l = 2 \), we delete the last element in \( A_{m-1} \) and get

\[
A_{m-2} = \left( 0, 1, m - 1, 2, m - 2, 3, m - 3, \ldots, \frac{m}{2} - 2, \frac{m}{2} + 2, \frac{m}{2} - 1 \right).
\]

We give examples in Fig. 1(i) \((m = 6, l = 1)\) and Fig. 1(ii) \((m = 8, l = 1)\).

Note that in Fig. 1(i), \( A_5 = (0, 1, 5, 2, 4) \), in which 3 is deleted and in Fig. 1(ii), \( A_7 = (0, 1, 7, 2, 6, 3, 5) \), in which 4 is deleted.

We generalize the above method and get

\[
A_{m-l} = \begin{cases} 
\left( 0, 1, m - 1, 2, m - 2, 3, m - 3, \ldots, \frac{m}{2} - \left( \frac{l}{2} + 2 \right), \frac{m}{2} \right) + \left( \frac{l}{2} + 1 \right), \frac{m}{2} + \left( \frac{l}{2} + 1 \right), \frac{m}{2} - \frac{l}{2} & \text{if } l \equiv 0 \pmod{2}; \\
\left( 0, 1, m - 1, 2, m - 2, 3, m - 3, \ldots, \frac{m}{2} - \left( \frac{l+1}{2} + 2 \right), \frac{m}{2} \right) + \left( \frac{l+1}{2} + 2 \right), \frac{m}{2} + \left( \frac{l+1}{2} + 1 \right), \frac{m}{2} - \frac{l+1}{2} & \text{if } l \equiv 1 \pmod{2}.
\end{cases}
\]

**Case 2:** When \( m \) is odd.

When \( l = 1 \), we can get \( A_{m-1} \) from \( Z_m \setminus [(m + 1)/4] \).

\[
A_{m-1} = (0, 1, m - 1, 2, m - 2, 3, \ldots, [(m + 1)/4] - 1, m - [(m + 1)/4] + 1, [(m + 1)/4] + 1, m - [(m + 1)/4], [(m + 1)/4] + 2, m - [(m + 1)/4] - 1, \ldots, (m + 1)/2 + 2, (m + 1)/2 - 1, (m + 1)/2 + 1, (m + 1)/2).
\]

Similarly, we can get \( A_{m-2} \) by deleting the last element in \( A_{m-1} \).

We give examples in Fig. 2(i) \((m = 7, l = 1)\) and Fig. 2(ii) \((m = 9, l = 1)\).

Note that in Fig. 2(i), \( A_6 = (0, 1, 6, 3, 5, 4) \), in which 2 is deleted and in Fig. 2(ii), \( A_8 = (0, 1, 8, 2, 7, 4, 6, 5) \), in which 3 is deleted.

We generalize the above method and get
Thus, we have

\[
A_{m-l} = \begin{cases} 
0,1,m-1,2,m-2,3,m-3,4,\ldots,\left\lceil \frac{m+1}{4} \right\rceil - 1, m \\
+1 - \left\lceil \frac{m+1}{4} \right\rceil, \left\lfloor \frac{m+1}{2} \right\rfloor + 1, m - \left\lceil \frac{m+1}{4} \right\rceil, \left\lfloor \frac{m+1}{2} \right\rfloor \\
+2, m - \left\lceil \frac{m+1}{4} \right\rceil - 1, \ldots, \left\lfloor \frac{l}{2} \right\rfloor + 1, \frac{m+1}{2} \\
+ \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right), \frac{m+1}{2} - \left\lfloor \frac{l}{2} \right\rfloor, \frac{m+1}{2} + \left\lfloor \frac{l}{2} \right\rfloor & \text{if } l \equiv 0 \pmod{2} \\
\end{cases}
\]

For two positive integers \(m, n, n \geq 7\), there exist maximal sets of \(m\) Hamilton cycles in \(D_n\) if and only if \([n/2] \leq m \leq n - 1\).

**Proof.** Suppose that \(m < [n/2]\), and let \(S\) be a set of \(m\) edge-disjoint directed Hamilton cycles in \(D_n\). Let \(G(S)\) be the graph consisting of the union of the \(m\) Hamilton cycles in \(S\). Then \(G(S)^c\) is an \(r\)-regular directed graph with \(r = n - m - 1 \geq [n/2]\) except the case \(n\) is odd, \(m = (n-1)/2\). So by Lemma 2.2, \(G(S)^c\) contains a Hamilton cycle and \(S\) is not a maximal set. When \(n\) is odd, \(m = (n-1)/2\), \(G(S)^c\) is an \((n-1)/2\)-regular directed graph of order \(n\), by Lemma 2.3, \(G(S)^c\) has at least one Hamilton cycle. Thus \(m \geq [n/2]\).

Now we shall show when \([n/2] \leq m \leq n - 2\), we can construct maximal sets of \(m\) Hamilton cycles in \(D_n\).

When integer \(l > 0\), let \(l = n - m - 1\). Otherwise let integer \(l = 0\). By Lemma 2.4, we can choose ordered set \(A_{m-l}\) and \(b_i\) \((1 \leq i \leq l)\) from \(Z_m\) and thus \(Z_m\) can be partitioned into \(A_{m-l}\) and everything else not in \(A_{m-l}\). That is, \(Z_m = A_{m-l} \cup \{b_1, b_2, \ldots, b_l\}\), where \(A_{m-l} = (a_1, a_2, \ldots, a_{m-l})\) and \(A_{m-l} \cap \{b_1, b_2, \ldots, b_l\} = \emptyset\).

Since \(D_n = D_{n-m} + D_{m,n-m} + D_m\), let \(V(D_{m,n-m}) = \{i_0| i \in Z_{n-m}\} \cup \{i|i \in Z_m\}\). Further, \(\{i|i \in Z_m\}\) can be partitioned as \(A_{m-l} \cup \{b_1, b_2, \ldots, b_l\}\). Now, we obtain a set \(S\) of directed Hamilton cycles with \(S = \{(0,0), (b_1 + j), 1_0, (b_2 + j), \ldots, (n - m - 2), (b_{m-m-l} + j), (n - m - 1), (A_{m-l} + j)\}| j \in Z_m\}\) from \(D_m + D_{m,n-m}\). Note that the entries of \(b_i + j|1 \leq i \leq n - m - 1\) and \(A_{m-l} + j\) are done modulo \(m\). Obviously, \(E(D_{m,n-m}) \subseteq E(S)\) and \(D_n - E(S)\) is disconnected. Thus \(S\) is a maximal set of Hamilton cycles in \(D_n\) when \([n/2] \leq m \leq n - 2\).

As for \(m = n - 1\), \(D_n\) can be decomposed into directed Hamilton cycles by Lemma 2.1. Thus, we conclude the proof.

**Example 1.** When \(n = 8, 4 \leq m \leq 7\), we give a construction of maximal sets of Hamilton cycles for \(m \in \{4, 5, 6\}\) and the case \(m = 7\) can be settled by applying Lemma 2.1:

- **\(m = 4\):** the set is \((0,0, j, 1_0, 1 + j, 2_0, 2 + j, 3_0, 3 + j)| j \in Z_4\).
- **\(m = 5\):** the set is \((0,0, j, j + 1, j + 4, 1_0, 2 + j, 2_0, 3 + j)| j \in Z_5\).
- **\(m = 6\):** the set is \((0,0, j, j + 1, j + 5, j + 2, j + 4, 1_0, 3 + j)| j \in Z_6\).
Example 2. When $n = 9$, $5 \leq m \leq 8$, we give a construction of maximal sets of Hamilton cycles for $m \in \{5, 6, 7\}$ and the case $m = 8$ can be settled by applying Lemma 2.1:

- $m = 5$, the set is $\{(0_0, j, j + 1, 1_0, j + 4, 2_0, 3 + j, 3_0, 2 + j) | j \in Z_5\}$.
- $m = 6$, the set is $\{(0_0, j, j + 1, j + 5, j + 2, 1_0, 3 + j, 2_0, 4 + j) | j \in Z_6\}$.
- $m = 7$, the set is $\{(0_0, j, j + 1, j + 6, j + 3, j + 5, j + 4, 1_0, 2 + j) | j \in Z_7\}$.

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