The Optimal Average Information Ratio of Secret-Sharing Schemes for the Access Structures Based on Bipartite Graphs and Unicycle Graphs

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Abstract A secret-sharing scheme is a method of distributing a secret among a set of participants in such a way that only qualified subsets of participants can recover the secret and the joint share of the participants in any unqualified subset is statistically independent of the secret. The collection of all qualified subsets is called the access structure of the scheme. In a graph-based access structure, each vertex of a given graph \( G \) represents a participant and each edge of \( G \) represents a minimal qualified subset. The average information ratio of a perfect secret-sharing scheme realizing a given access structure is the ratio of the average length of the shares given to the participants to the length of the secret. The infimum of the average information ratio of all possible perfect secret-sharing schemes realizing an access structure is called the optimal average information ratio of that access structure. In this paper, we derive bounds on the optimal average information ratio of the access structures based on general graphs and investigate the value of the ratio for bipartite graphs and unicycle graphs. We propose conditions under which the exact value of the optimal average information ratio can be determined. Consequently, we determine the exact values of the optimal average information ratio of some infinite classes of bipartite graphs and unicycle graphs. This extends previous results. We also provide a good bound on the optimal average information ratio for all unicycle graphs.

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1 Introduction

Motivated by the problem of secure information storage, secret-sharing schemes have found numerous applications in cryptography and distributed computing. A secret-sharing scheme involves a dealer who has a secret, a finite set $P$ of participants and a collection $\Gamma$ of subsets of $P$ called the access structure. Each subset in $\Gamma$ is a qualified subset. A secret-sharing scheme is a method by which the dealer distributes a secret among the participants in $P$ such that only the participants in a qualified subset can recover the secret and the joint share of the participants in any unqualified subset is statistically independent of the secret. An access structure is naturally required to be monotone, that is, any subset of $P$ containing a qualified subset must also be qualified. Therefore, an access structure $\Gamma$ is completely determined by the family of all its minimal subsets which is called the basis of $\Gamma$.

Shamir [20] and Blakley [3] independently introduced the first kind of secret-sharing schemes called the $(t,n)$-threshold schemes in 1979. In such a scheme, the basis of the access structure consists of all $t$-subsets of the participant set of size $n$. Their work has raised a great deal of interest in the research of many aspects of secret-sharing problems. Secret-sharing schemes for various access structures as well as many modified versions with additional capacities were widely studied. The information ratio and the average information ratio of secret-sharing schemes have been the main subjects of discussion. The information ratio of a secret-sharing scheme is the ratio of the maximum length (in bits) of the share given to a participant to the length of the secret, while the average information ratio of a secret-sharing scheme specifies the ratio of the average length of the shares given to the participants to the length of the secret. These ratios represent the maximum and the average number of bits a participant has to remember for each bit of the secret respectively. As opposed to them, some literatures use information rate and average information rate which are exactly the reciprocal of the information ratio and the average information ratio respectively. For lower storage and communication complexity, these ratios are expected to be as low as possible. The problem of constructing secret-sharing schemes with the lowest these ratios arose naturally. Given an access structure $\Gamma$, the infimum of the (average) information ratio of all possible secret-sharing schemes realizing this access structure $\Gamma$ is referred to as the optimal (average) information ratio of $\Gamma$.

In this paper, we consider graph-based access structures. Given a simple graph $G$, we let each vertex of $G$ represents a participant and each edge of $G$ represent a minimum qualified subset. That is, in the access structure based on
If an element $x$ in a secret-sharing scheme is defined as $(v, \{\zeta_v\}_{v \in V(G)})$, the average uncertainty associated with $X$ is the Shannon entropy, the information ratio of the scheme $\Sigma$ is the number of bits needed to represent the element in $X$ on it. For instance, “a secret-sharing scheme on $G$ is an ideal scheme for the access structure based on $G$” refers to “a secret-sharing scheme for the access structure based on $G$. Furthermore, the optimal information ratio $R(G)$ of $G$ and the optimal average information ratio $AR(G)$ of $G$ are the infimum of the information ratio $R_\Sigma$ and the average information ratio $AR_\Sigma$ over all possible secret-sharing schemes $\Sigma$ on $G$ respectively. It is well known that $R(G) \geq AR(G) \geq 1$ [10] and that $R(G) = 1$ if and only if $AR(G) = 1$. A secret-sharing scheme $\Sigma$ with the optimal ratio $R_\Sigma = 1$ or $AR_\Sigma = 1$ is called ideal. An access structure $G$ is ideal if there exists an ideal secret-sharing scheme on it. Due to the difficulty of determining the exact values of $R(G)$ and $AR(G)$, most known results give bounds on them [4–9], [11, 12, 16, 18], [21–23]. The exact values of the optimal average information ratio of most graphs of order no more than five and the optimal information ratio of most graphs of order no more than six have been determined [7, 16, 18]. Before 2007, apart from a specially defined class of graphs [4], the paths and cycles were the only infinite classes of graphs which have known optimal information ratio and optimal average information ratio. Csirmaz and Tardos’s [14] excellent work appeared in 2007. They determined the exact value of the optimal information ratio of all trees. In 2009, Csirmaz and Ligeti [13] gave the best result so far on the optimal information ratio of graphs by showing that $R(G) = 2 - 1/d$, where $d$ is the maximum degree of $G$, for any graph $G$ satisfying the following conditions: (i) every vertex has at most one neighbor of degree one; (ii) vertices of degree at least three are not connected by an edge; (iii) the girth of $G$ is at least six. In 2012, Lu and Fu [19] went on settling the exact values of the optimal average information ratio of all trees. Recently Beimel et al. [2] investigated the total share size of secret-sharing schemes realizing very dense graphs and showed excellent results. In this paper, we deal with the optimal average information ratio of bipartite graphs and unicycle graphs.
This paper is organized as follows. In Section 2, we recall basic definitions and restate some known results. In Section 3, we derive bounds on the optimal average information ratio for general graphs. In Section 4, we investigate the problem for bipartite graphs and propose conditions under which the value of the optimal average information ratio can be determined. As a consequence, we determine the exact value of this ratio for some infinite classes of bipartite graphs. This extends the result in [19]. In Section 5, we determine the value of the optimal average information ratio for some unicycle graphs and give a good bound on it for all unicycle graphs. A concluding remark will be given in the final section.

2 Preliminaries

In this section, we introduce some basic notions and important results for our discussion in the following sections. All graphs considered in this paper are simple graphs without loops and isolated vertices. The vertex number \(|V(G)|\) of a graph \(G\) is called the \textit{order} of \(G\) and the edge number \(|E(G)|\) of \(G\) is the \textit{size} of it.

Birkell and Davenport [8] have given a complete characterization of ideal graph-based access structures.

**Theorem 1** ([8]) Suppose that \(G\) is a connected graph. Then \(R(G) = AR(G) = 1\) if and only if \(G\) is a complete multipartite graph.

For a non-ideal access structure \(G\), we consider upper bounds and lower bounds on \(AR(G)\). The average information ratio of any secret-sharing scheme on \(G\) makes a natural upper bound on \(AR(G)\). Stinson’s decomposition construction [23] enables us to build up a secret-sharing scheme on a graph via its \textit{complete multipartite covering}. A complete multipartite covering of a graph \(G\) is a collection of complete multipartite subgraphs \(\Pi = \{G_1, G_2, \ldots, G_l\}\) of \(G\) such that each edge of \(G\) appears in at least one subgraph in this collection. The sum \(n_\Pi = \sum_{i=1}^{l} |V(G_i)|\) is called the \textit{vertex-number sum} of \(\Pi\).

**Theorem 2** ([23]) Suppose that \(\Pi = \{G_1, G_2, \ldots, G_l\}\) is a complete multipartite covering of a graph \(G\) of order \(n\). Then there exists a secret-sharing scheme \(\Sigma\) on \(G\) with average information ratio \(AR_\Sigma = n_\Pi/n\).

This theorem suggests that a complete multipartite covering of a graph with smaller vertex-number sum leads to a secret-sharing scheme on that graph with lower average information ratio. A complete multipartite covering with the least vertex-number sum is said to be \textit{optimal}. A good complete multipartite covering is our main tool to establish upper bounds on \(AR(G)\).

In the case that every subgraph in a complete multipartite covering is a star, this covering is also called a \textit{star covering}. If each edge of \(G\) appears in exactly one subgraph in a covering, then this covering is in fact a \textit{decomposition}. In the discussion of seeking star coverings with the least vertex-number sum, it
is sufficient to only consider star decompositions. A star decomposition works especially well for graphs of larger girth.

For the derivation of lower bounds on $AR(G)$, we use a method based on information theoretic approach [4–6], [10–14], [16]. Let $\Sigma$ be a secret-sharing scheme in which $\xi_s$ is the random variable of the secret and each $\xi_v$ is the random variable of the share of $v$, $v \in V(G)$. Define a real-valued function $f$ as $f(A) = H(\{\xi_v : v \in A\})/H(\xi_s)$ for each subset $A \subseteq V(G)$, where $H$ is the Shannon entropy. Then, $AR_{\Sigma} = \frac{1}{n} \sum_{v \in V(G)} f(v)$, where $n$ is the order of $G$.

Csirmaz et al. [13] defined a core of $G$ as a set of vertices $V_0 \subseteq V(G)$ satisfying that (i) $V_0$ induces a connected subgraph in $G$; (ii) each vertex $v \in V_0$ has a designated outside neighbor which is defined as a neighbor $\bar{v}$ of $v$ that is outside $V_0$ and is not adjacent to any other vertex in $V_0$, and (iii) $\{v\bar{v} : v \in V_0\}$ is an independent set in $G$. Using the idea of a core, they show the following theorem.

**Theorem 3** ([14]) Let $V_0$ be a core of a graph $G$. If $f$ is defined as above, then $\sum_{v \in V_0} f(v) \geq 2|V_0| - 1$.

Based on the facts introduced in this section, we shall derive lower bounds and upper bounds on $AR(G)$ in the following section.

### 3 Lower bound and upper bound on $AR(G)$

For deriving lower bounds on $AR(G)$, we define a core cluster of $G$ of size $k$ as a partition $\mathcal{C} = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ such that each $V_i$, $i \in \{1, 2, \ldots, k\}$, is a core of $G$. We denote the minimum size of a core cluster of $G$ as $\tilde{c}(G)$. A core cluster of size $\tilde{c}(G)$ is called an optimal core cluster of $G$. With this definition, we have a lower bound on $AR(G)$ for general graphs.

**Theorem 4** If $G$ is a graph of order $n$, then $AR(G) \geq 2 - \frac{\tilde{c}(G)}{n}$.

**Proof** Let $\mathcal{C} = \{V_1, V_2, \ldots, V_k\}$ be a core cluster of $G$. If $\Sigma$ is a secret-sharing scheme on $G$, then we have $\sum_{v \in V(G)} f(v) = \sum_{i=1}^{k} \sum_{v \in V_i} f(v) \geq \sum_{i=1}^{k} \left(2|V_i| - 1\right) = 2|V(G)| - k$, where $f$ is the function defined in Section 2. Thus $AR_{\Sigma} \geq \frac{1}{n}(2n - k) = 2 - \frac{k}{n}$. Since this inequality holds for any secret-sharing scheme on $G$, we have the result. □

This proof is quite similar to the one used to show the similar result for trees in [19]. Note that our definition of a core cluster is slightly different from the one used in [19] in which a core cluster is a partition of the set of vertices whose degree are at least two instead of a partition of $V(G)$. Using our definition, the main result in [19] can be written in a neater way as follows.

**Theorem 5** ([19]) If $T$ is a nontrivial tree of order $n$, then $AR(T) = 2 - \frac{\tilde{c}(T)}{n}$.

Combining Theorem 2 and Theorem 4, we have the following bound on $AR(G)$ for general graphs.
Theorem 6 If $G$ is a graph of order $n$ and $\Pi$ is a complete multipartite covering of $G$ with vertex-number sum $n_{\Pi}$, then $2 - \frac{\bar{\chi}(G)}{n} \leq AR(G) \leq n_{\Pi}$. If there exists a complete multipartite covering (decomposition) $\Pi$ of $G$ with vertex-number sum $n_{\Pi}$ and a core cluster $\mathcal{C}$ of $G$ of size $k$ such that $2|V(G)| - k = n_{\Pi}$, then we called the graph $G$ realizable and therefore $AR(G) = 2 - \frac{\bar{\chi}(G)}{n}$.

Next, we consider upper bounds on $AR(G)$ derived from star decompositions of $G$. Any star decomposition $\Pi = \{S_1, S_2, \ldots, S_l\}$ of a graph $G$ naturally induces an orientation on $G$ in the following way. For $i \in \{1, 2, \ldots, l\}$, we direct each edge of the star $S_i$ from the center toward the leaf. Let us denote the resulting directed graph as $G_\Pi$ and the number of the sink vertices in $G_\Pi$ as $s_\Pi$. Since the set of sink vertices in $G_\Pi$ is an independent set in $G$, we have $s_\Pi \leq \alpha(G)$. Let us introduce the following lower bound on the vertex-number sum of a star decomposition (covering) of a general graph.

Lemma 1 The vertex-number sum of any star covering of a graph $G$ is at least $|V(G)| + |E(G)| - \alpha(G)$.

Proof Considering star decompositions is sufficient. Let $\Pi = \{S_1, S_2, \ldots, S_l\}$ be a star decomposition of $G$ and $G_\Pi$ be the directed graph induced by $\Pi$. Since we may assume that each vertex is the center of at most one star in this decomposition, we have $l = |V(G)| - s_\Pi$. Now, the vertex-number sum $n_{\Pi}$ of $\Pi$ can be evaluated as follows.

$$n_{\Pi} = \sum_{i=1}^{l} |V(S_i)| = l + \sum_{i=1}^{l} |\{\text{leaves of } S_i\}| = (|V(G)| - s_{\pi}) + |E(G)|$$

$$\geq |V(G)| + |E(G)| - \alpha(G)$$

On the other hand, given an independent set $L$ in $G$, we are able to construct a star decomposition $\Pi$ such that the set of sink vertices in the directed graph $G_\Pi$ induced by $\Pi$ is exactly the given independent set $L$.

Proposition 1 For any independent set $L$ in $G$, there exists a star decomposition $\Pi$ of $G$ such that $L$ is the set of all sink vertices in $G_\Pi$ and $n_{\Pi} = |V(G)| + |E(G)| - |L|$.

Proof For the given independent set $L$, we define an orientation on $G$. First, since each edge is incident to at most one vertex in $L$, we can direct all incident edges of any $v \in L$ toward $v$. Second, for an endvertex $u$ of a directed edge $uv$, if there is any undirected edge incident to $u$, then $u$ must be the tail of the edge $uv$. Let us direct all undirected edges incident to $u$ toward $u$. This makes $u$ a nonsink vertex. As $G$ is finite, all edges will eventually be directed by repeatedly doing the second step. Note that in the resulting directed graph $G$, the set of all sink vertices is exactly $L$. Besides, if we let $S_u$ for each $u \in V(G) \setminus L$, be the star whose center is $u$ and whose leaves are the heads of
the edges which share the same tail \( u \). Then \( II = \{S_u | u \in V(G) \setminus L \} \) is the required star decomposition with \( n_H = |V(G)| + |E(G)| - |L| \) and the directed graph \( G \) is exactly \( G_H \).

Therefore, the star decomposition is optimal when the given independent set is maximum.

**Corollary 7** Any graph \( G \) has an optimal star decomposition whose vertex-number sum equals \( |V(G)| + |E(G)| - \alpha(G) \).

Using this result, the bound on \( AR(G) \) in Theorem 6 can be written as follows.

**Theorem 8** If \( G \) is a graph of order \( n \) and size \( m \), then \( 2 - \frac{\gamma'(G)}{n} \leq AR(G) \leq \frac{n + m - \alpha(G)}{n} \).

In the directed graph \( G_H \) induced by some star decomposition \( II \) of \( G \), the in-degree of \( v \) is denoted as \( d_{G_H}(v) \). The subgraph of \( G_H \) induced by a subset \( V' \) of the vertex set of \( G \) is written as \( G_H[V'] \). Let \( \mathcal{C} = \{V_1, V_2, \ldots, V_k\} \) be a core cluster of \( G \), we use \( m_i' \) to denote the number of edges of \( G_H \) whose heads are in \( V_i \) and whose tails are not in \( V_i \), for each \( i = 1, 2, \ldots, k \). In addition, the minimum length of a cycle in \( G \) is referred to as the girth of \( G \) and is denoted as \( \text{girth}(G) \). In order to have the upper bound and the lower bound on \( AR(G) \) in Theorem 8 meet, we investigate the relationship between an optimal core cluster and an optimal star decomposition of \( G \) in the next theorem.

**Theorem 9** If \( \text{girth}(G) \geq 5 \), then \( G \) is realizable if and only if there exists a star decomposition \( II \) and a core cluster \( \mathcal{C} \) of \( G \) such that each core \( V_i \) in \( \mathcal{C} \) satisfies the following conditions:

(i) \( V_i \) induces a tree in \( G \);
(ii) \( d_{G_H[V_i]}(v) = d_G(v) - 1 \) for each sink vertex \( v \in V_i \) of \( G_H \) and \( d_{G_H[V_i]}(u) = d_{G_H}(u) \) for each nonsink vertex \( u \in V_i \) of \( G_H \).

**Proof** Let \( II \) be a complete multipartite covering and \( \mathcal{C} = \{V_1, V_2, \ldots, V_k\} \) be a core cluster of \( G \). From previous discussion, we know that \( 2|V(G)| - k \leq |V(G)|AR(G) \leq n_H \). Since \( \text{girth}(G) \geq 5 \), we may assume that \( II \) is a star decomposition and then we have

\[
\sum_{i=1}^{k} (2|V_i| - 1) \leq |V(G)| + |E(G)| - s_H
\]

\[
= \sum_{i=1}^{k} |V_i| + \sum_{i=1}^{k} (|E(G_H[V_i])| + m_i') - \sum_{i=1}^{k} |\text{sinks of } G_H \text{ in } V_i|
\]

\[
= \sum_{i=1}^{k} (|V_i| + |E(G_H[V_i])| + m_i' - |\text{sinks of } G_H \text{ in } V_i|)
\]
Corollary 10 If there exists a star decomposition $\Pi$ and a core cluster $\mathcal{C}$ of $G$ satisfying the conditions stated in Theorem 9, then $G$ is realizable and $AR(G) = \frac{|V(G)| + |E(G)| - \alpha(G)}{|V(G)|}$.

The discussion in the proof of Theorem 9 also leads to a bound on $AR(G)$ as follows.

Corollary 11 If $\Pi$ is an optimal star decomposition of $G$ and $\mathcal{C} = \{V_1, \ldots, V_k\}$ is a core cluster of $G$, then

$$\frac{|V(G)| + |E(G)| - \alpha(G) - \beta(\Pi, \mathcal{C})}{|V(G)|} \leq AR(G) \leq \frac{|V(G)| + |E(G)| - \alpha(G)}{|V(G)|},$$

where $\beta(\Pi, \mathcal{C}) = \sum_{i=1}^{k} (|E(G_{\Pi}[V_i])| - |V_i| + 1 + m_{V_i} - |\text{sinks of } G_{\Pi} \text{ in } V_i|)$.

Note that condition (ii) in Theorem 9 requires that there is no edge going from outside $G_{\Pi}[V_i]$ into a nonsink vertex of $G_{\Pi}$ in $V_i$. We have the following observations.

Corollary 12 If $G$ is realizable with $girth(G) \geq 5$ and both endvertices of an edge $uv \in E(G)$ are centers of some stars in an optimal star decomposition of $G$, then $u$ and $v$ must be in the same core in any optimal core cluster of $G$.

Corollary 13 If $girth(G) \geq 5$ and there exists a maximum independent set $L$ in $G$ such that $G - L$ contains cycles, then $G$ is not realizable.

Proof Consider the optimal star decomposition $\Pi$ from Corollary 7. Since the endvertices of any edge in $G - L$ are both centers of some stars in $\Pi$, they must be in the same core in any optimal core cluster of $G$. The vertices of any component in $G - L$ are in the same core in any optimal core cluster of $G$. Therefore, $G$ is not realizable because there is a core inducing cycles. □
4 Bipartite Graphs

Now, let us consider bipartite graphs. Let $G = (X, Y)$ be a bipartite graph and $L$ be a maximum independent set of $G$, then $K = V(G) \setminus L$ is a minimum vertex cover. By König Theorem, there exists a set of edges $E_L = E_1 \cup E_2$ where $E_1$ is between $L \cap X$ and $K \cap Y$, and $E_2$ is between $L \cap Y$ and $K \cap X$ such that each vertex in $L \cap X$ is on exactly one edge from $E_1$ and each vertex in $K \cap Y$ is on some edge from $E_1$, and each vertex in $L \cap Y$ is on exactly one edge from $E_2$ and each vertex in $K \cap X$ is on some edge from $E_2$. We call the set $E_L$ a fan associated with the maximum independent set $L$. In the next lemma, we will show that if the vertex set of each component in $G - E_L$ is a core of $G$, then the core cluster $\{V(H)|H$ is a component in $G - E_L\}$ naturally satisfies some properties required for $G$ being realizable.

**Lemma 2** Let $E_L$ be a fan associated with some maximum independent set $L$ in a bipartite graph $G$ and $H$ be a component in $G - E_L$. Let $\Pi$ be an optimal star decomposition induced by $L$ from Corollary 7. If $V(H)$ is a core of $G$, then the following statements are true.

(i) The designated outside neighbor of each $v \in L \cap V(H)$ must be the unique neighbor $\bar{v}$ of $v$ with $\bar{v} \in E_L$ and the one of each $u \in V(H) \setminus L$ is a neighbor $\bar{u} \in L$ of $u$ with $u\bar{u} \in E_L$;

(ii) $d_{G_H(V(H))}(v) = d_G(v) - 1$ for each sink vertex $v \in V(H)$ of $G_H$ and $d_{G_H(\bar{V}(H))}(u) = d_{G_H}(u)$ for each nonsink vertex $u \in V(H)$ of $G_H$;

(iii) $m_{V(H)} = |\{\text{sink vertices of } G_H \text{ in } V(H)\}|$.

**Proof** Since $V(H)$ is a core, each vertex $v$ in $V(H)$ has a designated outside neighbor which is not in $V(H)$. On the other hand, the only neighbors of $v$ that are possibly outside $H$ are the other endvertices of the incident edges of $v$ in $E_L$ and each vertex in $L \cap V(H)$ has at most one such neighbor. Therefore statement (i) is obviously true and $d_{G_H(V(H))}(v) = d_G(v) - 1$ for each sink vertex $v \in V(H)$ of $G_H$. According to the orientation induced by $\Pi$, any edge whose head is $u \in V(H) \setminus L$ connects $u$ to a vertex $u' \notin L$ and therefore $uu' \notin E_L$. Hence $d_{G_H(\bar{V}(H))}(u) = d_{G_H}(u)$ for each nonsink vertex $u \in V(H)$ of $G_H$ and $m_{V(H)} = |\{\text{sink vertices of } G_H \text{ in } V(H)\}|$. 

From this discussion and Corollary 11, we have the following bound on $AR(G)$ for bipartite graphs.

**Corollary 14** If a bipartite graph $G$ of size $m$ and order $n$ has a fan $E_L$ associated with some maximum independent set $L$ such that the vertex set of each component in $G - E_L$ is a core of $G$, then

$$\frac{n + m - \alpha(G) - \sum_{i=1}^{k} (|E(G[V_i])| - |V_i| + 1)}{n} \leq AR(G) \leq \frac{n + m - \alpha(G)}{n},$$

where $V_i = V(H_i)$ and the $H_i$’s, $i = 1, \ldots, k$, are the components in $G - E_L$. 
In this situation, if the vertex set of each component induces no cycles in $G$, then $G$ is realizable.

**Corollary 15** If a bipartite graph $G$ has a fan $E_{\ell}(L)$ associated with some maximum independent set $L$ such that the vertex set of each component in $G - E_{\ell}(L)$ is a core of $G$ and induces no cycles in $G$, then $\text{AR}(G) = \frac{|V(G)| + |E(G)| - \alpha(G)}{|V(G)|}$.

Let $H$ be a subgraph of $G$. For each $u \in V(H)$, we let $e_u$ be an edge of $G$ which joins $u$ to a vertex not in $V(H)$. If we denote the set $\{e_u | u \in V(H)\}$ as $E'$, then the extended graph $H + E'$ of $H$ is defined as the graph obtained by adding to $H$ all edges in $E'$. That is, $V(H + E') = V(H) \cup \{w | uw \in E' \text{ where } u \in V(H) \text{ and } w \notin V(H)\}$ and $E(H + E') = E(H) \cup E'$.

**Theorem 16** Suppose that $E_{\ell}(L)$ is a fan associated with some maximum independent set $L$ in a bipartite graph $G$. For each component $H$ in $G - E_{\ell}(L)$, let $e_v$ be an incident edge of $v \in V(H)$ in $E_{\ell}(L)$ and $E_H^+ = \{e_v | v \in V(H)\}$. If the vertex set of the extended graph $H + E_H^+$ induces no cycles in $G$, then $\text{AR}(G) = \frac{|V(G)| + |E(G)| - \alpha(G)}{|V(G)|}$.

**Proof** By Corollary 15, it suffices to show that the vertex set of each component in $G - E_{\ell}(L)$ is a core of $G$. Since any component $H$ induces no cycles in $G$, the endvertices of each edge in $E_{\ell}(L)$ lie in different components in $G - E_{\ell}(L)$. For each vertex $v \in V(H)$, we choose the designated outside neighbor of $v$ as the endvertex $\bar{v}$ of $vv \in E_H^+$. Since $H + E_H^+$ induces no cycles, any two distinct vertices of $H$ can not share the same designated outside neighbor and the designated outside neighbors of any two distinct vertices of $H$ are of distance at least two. We conclude that the set $V(H)$ is a core of $G$.

It seems very unlikely to give a complete characterization of the graphs which satisfy the criterion in Theorem 16. In what follows, we propose some infinite classes of such graphs. Recall that the maximum distance of two vertices in a graph $H$ is referred to as the diameter of $H$ and is written as $\text{diam}(H)$.

**Theorem 17** If a bipartite graph $G$ has a fan $E_{\ell}(L)$ associated with some maximum independent set $L$ such that $\text{diam}(H) \leq \text{girth}(G) - 4$ for any component $H$ in $G - E_{\ell}(L)$, then $\text{AR}(G) = \frac{|V(G)| + |E(G)| - \alpha(G)}{|V(G)|}$.

**Corollary 18** For any nontrivial tree $T$ of order $n$, $\text{AR}(T) = 2 - \frac{(\alpha(T) + 1)}{n}$ and $\bar{e}^*(T) = \alpha(T) + 1$.

**Proof** The criterion in Theorem 17 is satisfied by any fan of a tree. Therefore we have the value of $\text{AR}(T)$ as

$$\text{AR}(T) = \frac{n + |E(T)| - \alpha(T)}{n} = \frac{n + (n - 1) - \alpha(G)}{n} = 2 - \frac{(\alpha(T) + 1)}{n}$$

and $\bar{e}^*(T) = \alpha(T) + 1$. □
The first part of Corollary 18 is equivalent to the main result in [19] which is stated in Theorem 5 using our notation and definition. Our approach in this paper gives a more clear description of the value $\tilde{c}^*(T)$. To introduce another class of realizable bipartite graphs, we define the distance $d(e_1, e_2)$ of two distinct edges $e_1 = u_1u_2$ and $e_2 = v_1v_2$ as $d(e_1, e_2) = \min\{d(u_i, v_j) | i, j \in \{1, 2\}\}.$

**Theorem 19** If a bipartite graph $G$ has a fan $E_{i(L)}$ associated with some maximum independent set $L$ such that each cycle of $G$ contains two edges from $E_{i(L)}$ which are of distance at least two, then $AR(G) = \frac{|V(G)| + |E(G)| - \alpha(G)}{|V(G)|}$.

**Proof** For each component $H$ in $G - E_{i(L)}$, we consider any cycle $C = (u_1, u_2, \cdots, u_k)$ of $G$ with $u_1 \in V(H)$. Let $a$ be the smallest index in $\{1, 2, \cdots, k\}$ such that $u_a u_{a+1}$ is in $E_{i(L)}$, then $1 \leq a \leq k - 3$. According to the assumption, any path from $u_1$ to $u_i$, $i \in \{a + 1, a + 2, a + 3\}$, not passing the edge $u_a u_{a+1}$ has another edge in $E_{i(L)}$. This edge is not incident to $u_{a+2}$ and is removed from $G$ while constructing $G - E_{i(L)}$. We therefore have that $u_{a+1}, u_{a+2}$ and $u_{a+3}$ are not vertices of $H$ and $u_{a+2}$ does not belong to any extended graph of $H$. Since this is true for any cycle containing a vertex of $H$, we conclude that the vertex set of any extended graph of $H$ induces no cycles in $G$. The result follows by Theorem 16.

A $k$-vertex of $G$ is a vertex whose degree is $k$. Another infinite class of realizable bipartite graphs is introduced in the next corollary.

**Corollary 20** Let $G$ be a bipartite graph. If every cycle of $G$ contains two 2-vertices of distance at least four, then $AR(G) = \frac{|V(G)| + |E(G)| - \alpha(G)}{|V(G)|}$.

**5 Unicycle Graphs**

Let $G$ be a unicycle graph. We denote the unique cycle of $G$ as $C_G = (v_1, v_2, \cdots, v_k)$. For simplicity, in the discussion in this section, $v_t$ always represents $v_i$, where $t \equiv i \pmod{k}$ and $1 \leq t \leq k$. $G$ is called an even unicycle graph if $k$ is even and an odd unicycle graph if $k$ is odd. Removing the edges of the cycle $C_G$ from $G$ results in $k$ trees, among which the tree containing $v_t$ is written as $T_t$. In addition, if there exists a maximum independent set in $T_t$ which does not contain $v_t$, then $T_t$ is called a Type I subtree of $G$. Otherwise, $T_t$ is of Type II. A Type II subtree may be trivial. For unicycle graphs, the bounds on $AR(G)$ can be written from Theorem 8 as follows.

**Theorem 21** If $G$ is a unicycle graph of order $n$, then $2 - \frac{\tilde{\epsilon}^*(G)}{n} \leq AR(G) \leq 2 - \frac{\alpha(G)}{n}$.

**Corollary 22** If $G$ is a unicycle graph and $girth(G) \geq 5$, then $G$ is realizable if and only if $\alpha(G) = \tilde{\epsilon}^*(G)$.

Let us introduce some properties of Type I and Type II subtrees of $G$ first.
Lemma 3 Let $G$ be a unicycle graph with $C_G = (v_1, v_2, \ldots, v_k)$. If $T_i$ is a Type I subtree of $G$ and $u$ is $v_{i-1}$ or $v_{i+1}$, then

(i) $\alpha(T_i + u) = \alpha(T_i) + 1$ and $\tilde{c}(T_i + u) = \alpha(T_i) + 2$;

(ii) there is an optimal core cluster $\mathcal{C}' = \{V_1', V_2', \ldots, V_{\alpha(T_i)+2}''\}$ of $T_i + u$ with $V'_1 = \{u\}$ such that the designated outside neighbor of $v_i$ is not $u$.

Proof For any independent set $L'$ in $T_i + u$, $L' \backslash \{u\}$ is independent in $T_i$. We have $\alpha(T_i + u) - 1 \leq |L' \backslash \{u\}| \leq \alpha(T_i)$, or $\alpha(T_i + u) \leq \alpha(T_i) + 1$. On the other hand, since $T_i$ is of Type I, there is a maximum independent set $L$ in $T_i$ which does not contain $v_i$, then $L \cup \{u\}$ is an independent set in $T_i + u$. Hence, $\alpha(T_i + u) = \alpha(T_i) + 1$ and the set $L \cup \{u\}$ is a maximum independent set in $T_i + u$. The maximum independent set $L$ in $T_i$ must contain at least one neighbor $w$ of $v_i$ in $T_i$ such that $v_iw$ is in a fan of $T_i$ associated with $L$. The designated outside neighbor of $v_i$ in the core cluster of $T_i$ constructed from the fan can be chosen as $w$. Now, by choosing the designated outside neighbor of $u$ as $v_i$, $\mathcal{C} \cup \{\{u\}\}$ is a core cluster of $T_i + u$. Since $T_i + u$ is a nontrivial tree, we know from Corollary 18 that $\tilde{c}(T_i + u) = \alpha(T_i + u) + 1 = \alpha(T_i) + 2 = \tilde{c}(T_i) + 1$. Thus $\mathcal{C} \cup \{\{u\}\}$ is an optimal core cluster of $T_i + u$ satisfies condition (ii). \qed

Lemma 4 Let $G$ be a unicycle graph with $C_G = (v_1, v_2, \ldots, v_k)$. If $T_i$ is a Type II subtree of $G$ and $u$ is $v_{i-1}$ or $v_{i+1}$, then

(i) $\alpha(T_i + u) = \alpha(T_i) + 1$ and $\tilde{c}(T_i + u) = \alpha(T_i) + 1$;

(ii) there is an optimal core cluster $\mathcal{C}' = \{V_1', V_2', \ldots, V_{\alpha(T_i)+1}''\}$ of $T_i + u$ with $V'_1 = \{u\}$ such that $v_i$ and $u$ are the designated outside neighbors of each other.

Proof Since $L' \backslash \{u\}$ is independent in $T_i$ for any independent set $L'$ in $T_i + u$, we have $\alpha(T_i + u) - 1 \leq |L' \backslash \{u\}| \leq \alpha(T_i)$, that is, $\alpha(T_i + u) \leq \alpha(T_i) + 1$. Suppose that $\alpha(T_i + u) = \alpha(T_i) + 1$, than there exists a maximum independent set $L'$ in $T_i + u$ containing the vertex $v_i$. In this case, $L' \backslash \{u\}$ is a maximum independent set in $T_i$ of size $\alpha(T_i)$ not containing the vertex $v_i$. This contradicts to the fact that $T_i$ is a Type II subtree of $G$. Therefore we have $\alpha(T_i + u) = \alpha(T_i)$. In addition, $\tilde{c}(T_i + u) = \alpha(T_i + u) + 1 = \alpha(T_i) + 1$. Since any fan of $T_i + u$ associated with a maximum independent set contains the edge $uv_i$, the optimal core cluster of $T_i + u$ constructed from the fan satisfies condition (ii). \qed

For a unicycle graph $G$, we define a suitable maximum independent set in $G$ as a maximum independent set $L$ in $G$ satisfying the following requirements.

(a) $L$ contains the least number of vertices of $C_G = (v_1, v_2, \ldots, v_k)$ among all maximum independent sets in $G$.

(b) For $v_j \not\in L$, if $V(T_j) \cap L$ is not a maximum independent set in $T_j$, then $(V(T_j) \cap L) \cup \{v_j\}$ is, $j \in \{1, 2, \ldots, k\}$.

(c) If $v_i \in L$, then $(L \backslash \{v_i\}) \cup \{v_{i-1}\}$ is not independent in $G$, $i \in \{1, 2, \ldots, k\}$.

Proposition 2 If $G$ is an odd unicycle graph with at least one Type I subtree or an even unicycle graph, then $G$ has a suitable maximum independent set.
Proof Let $L$ be a maximum independent set containing the least number of vertices of $C_G$ among all maximum independent sets in $G$. We adjust it to make a suitable one. For $v_j \notin L$, if $V(T_j) \cap L$ is not a maximum independent set in $T_j$, then there is a maximum independent set $L_{T_j}$ in $T_j$ containing $v_j$ such that $|L_{T_j}| > |V(T_j) \cap L|$. Now, replacing $V(T_j) \cap L$ with $L_{T_j} \setminus \{v_j\}$ in $L$, we have a maximum independent set satisfying the second requirement in the definition. Next, for each $i = 1, 2, \cdots$, we successively check that whether the situation "$v_i \notin L$ and $(L \setminus \{v_i\}) \cup \{v_{i-1}\}$ is a maximum independent set in $G$" occurs or not. If so, we replace $v_i$ with $v_{i-1}$ in $L$. We repeat this checking and replacing process for each $i = 1, 2, \cdots$ until no vertex in $L \cap V(G)$ can be shifted forward. This process will eventually stop if $G$ is an odd unicycle graph with at least one Type I subtree or an even unicycle graph. □

One can see from the proof that if $G$ is an odd unicycle graph without Type I subtrees, then a maximum independent set satisfying the first and second requirements does exist. It is the third requirement that can not be made because the process of shifting forward vertices in $L \cap V(G)$ may not stop. We have the following two observations about a suitable maximum independent set $L$.

**Lemma 5** If $G$ is an odd unicycle graph with at least one Type I subtree or an even unicycle graph whose cycle is $C_G = (v_1, v_2, \cdots, v_k)$, then any suitable maximum independent set $L$ in $G$ has the following properties.

(i) If $v_i \notin L$, then $v_{i-2} \in L$ or there is a neighbor of $v_{i-1}$ in $V(T_{i-1}) \cap L$.

(ii) If $v_j \notin L$, then $v_{j-1} \in L$ or $V(T_j) \cap L$ is a maximum independent set in $T_j$.

Proof The first property comes directly from requirement (c) in the definition. Let us show that the second property is true as well. If $v_j \notin L$ and $V(T_j) \cap L$ is not a maximum independent set in $T_j$, then $(V(T_j) \cap L) \cup \{v_j\}$ is, by requirement (b). In this case, if $v_{j+1}$ is not in $L$, then $v_{j+1}$ must not in $L$ either for otherwise $(L \setminus \{v_{j+1}\}) \cup \{v_j\}$ would be a maximum independent set in $G$, contradicting to requirement (c). Since both $v_{j-1}$ and $v_{j+1}$ are not in $L$, $L \cup \{v_j\}$ becomes a larger independent set in $G$, which also leads to a contradiction. Therefore, we conclude that if $v_j \notin L$ and $V(T_j) \cap L$ is not a maximum independent set in $T_j$, then $v_{j-1}$ must be in $L$. □

**Lemma 6** Let $G$ be an odd unicycle graph with at least one Type I subtree or an even unicycle graph whose cycle is $C_G = (v_1, v_2, \cdots, v_k)$. If $L$ is a suitable maximum independent set in $G$, then there exists a set $E'$ of edges of $G$ such that the following conditions are satisfied.

(i) Each edge in $E'$ connecting a vertex in $L$ and a vertex not in $L$.

(ii) Each vertex in $L$ is on exactly one edge from $E'$ and each vertex in $V(G) \setminus L$ is on some edge from $E'$.

(iii) $v_iv_{i+1} \in E'$ for any $v_i \in L$.

(iv) If $v_j \notin L$, then $v_{j-1}v_j \in E'$ where $v_{j-1} \in L$, or $v_jw \in E'$ where $w \in L \cap V(T_j)$. 

Proof For the given independent set \( L \), we construct the required set \( E' \) in the following way. First, if \( v_i \in L \), then \( T_i \) is of Type II and therefore, by Lemma 4, \( L \cap V(T_i) \) is a maximum independent set in \( T_i + v_{i+1} \). Hence, any fan \( F_i \) of \( T_i + v_{i+1} \) associated with \( L \cap V(T_i) \) must contain the edge \( v_i v_{i+1} \). Second, for \( v_j \notin L \), if \( L \cap V(T_j) \) is a maximum independent set in \( T_j \), then we let \( F_j \) be any fan of \( T_j \) associated with \( L \cap V(T_j) \). \( F_j \) certainly contains an edge \( v_j w \) with \( w \in L \cap V(T_j) \). If \( L \cap V(T_j) \) is not a maximum independent set in \( T_j \), then \( (L \cap V(T_j)) \cup \{ v_j \} \) is and \( T_j \) is of Type II. In this case, Lemma 5 asserts that \( v_{j-1} \) must be in \( L \) and Lemma 4 guarantees that \( (L \cap V(T_j)) \cup \{ v_j \} \) is also a maximum independent set in \( T_j + v_{j-1} \). Thus any fan \( F_j \) of \( T_j + v_{j-1} \) associated with \( (L \cap V(T_j)) \cup \{ v_j \} \) contains the edge \( v_{j-1} v_j \) which is also an edge in \( F_{j-1} \). Now, we let \( E' = \bigcup_{i=1}^{k} F_i \). Since each \( F_i \) is a fan of a tree; each \( v_i \in L \) is on exactly one edge from \( E' \) and each \( v_j \notin L \) is on some edge from \( E' \) joining \( v_j \) to some vertex in \( L \). \( E' \) has the required properties.

Note the set \( E' \) is in fact a fan \( E_{(L)} \) associated with \( L \) when \( G \) is an even unicycle graph. Therefore we also call it a fan associated with the suitable maximum independent set \( L \) when \( G \) is an odd unicycle graph with at least one Type I subtree and denote it as \( E_{(L)} \) in what follows. Now, a bound on \( AR(G) \) for a special case follows immediately.

**Proposition 3** Let \( G \) be a unicycle graph of order \( n \) with the unique cycle \( C_G \). If \( G \) has a maximum independent set \( L \) such that \( V(C_G) \cap L = \emptyset \), then

\[
2 - \frac{\alpha(G) + 1}{n} \leq AR(G) \leq 2 - \frac{\alpha(G)}{n}.
\]

**Proof** Let \( C_G = (v_1, v_2, \cdots, v_k) \). In this case, each subtree \( T_j \) must be of Type I and \( L \cap V(T_j) \) is a maximum independent set in \( T_j \). Hence a fan \( E_{(L)} \) which has the properties in Lemma 6 contains an edge \( v_j w \) with \( w \in L \cap V(T_j) \) for each \( j = 1, 2, \cdots, k \). Now, each \( V(H) \), where \( H \) is a component in \( G - E_{(L)} \), is a core of \( G \) because the designated outside neighbor \( \bar{u} \) of any vertex \( u \in V(H) \) can be chosen as the other endvertex of an edge \( uu' \in E_{(L)} \) and all \( \bar{u}' \)’s are independent. Since there are \( \alpha(G) + 1 \) components in \( G - E_{(L)} \), \( \bar{c}^*(G) \leq \alpha(G) + 1 \) and the result follows by Theorem 21.

The next result has been shown to be true for bipartite graphs in Section 4, we now show that it holds as well for odd unicycle graphs which have at least one Type I subtree.

**Lemma 7** Suppose that \( G \) is an odd unicycle graph with at least one Type I subtree or an even unicycle graph whose cycle is \( C_G = (v_1, v_2, \cdots, v_k) \) and \( E_{(L)} \) is a fan obtained from Lemma 6 associated with a suitable maximum independent set \( L \) in \( G \). Let \( \Pi \) be an optimal star decomposition of \( G \) from Corollary 7 and \( H \) be a component in \( G - E_{(L)} \). If, for each \( u \in V(H) \), there exists a neighbor \( \bar{u} \) of \( u \) with \( \bar{u}u \in E_{(L)} \) such that \( \bar{u} \notin V(H) \), then

(i) \( d^-_{G[U(H)]}(v) = d_G(v) - 1 \) for each sink vertex \( v \in V(H) \) of \( G_\Pi \) and \( d^-_{G[H]}(u) = d^-_{G}[u] \) for each nonsink vertex \( u \in V(H) \) of \( G_\Pi \), and
(ii) $m_{V(H)}^{-} = |\{\text{sink vertices of } G_H \text{ in } V(H)\}|$.

Proof Since each $v \in L \cap V(H)$ is on exactly one edge $vw$ in $E(L)$ and $v \notin V(H)$, we have $d_{G_H[V(H)]}(v) = d_G(v) - 1$. According to the orientation induced by $H$, each edge of $G_H$ going into a nonsink vertex $u \in V(H)$ of $G_H$ does not belong to $E(L)$ because it connects two nonsink vertices of $G_H$. So we have $d_{G_H[V(H)]}(u) = d_{G_H}(u)$ for each nonsink vertex $u \in V(H)$ of $G_H$ and therefore $m_{V(H)}^{-} = |\{\text{sink vertices of } G_H \text{ in } V(H)\}|$.

Next, we propose some realizable unicycle graphs.

Theorem 23 Let $G$ be an odd unicycle graph with at least one Type I subtree or an even unicycle graph whose cycle is $C_G = (v_1, v_2, \cdots, v_k)$ and $L$ be a suitable maximum independent set in $G$. Then $G$ is realizable if one of the following statements is true.

(i) $|L \cap V(C_G)| \geq 3$;
(ii) $L \cap V(C_G) = \{v_i, v_j\}$ and $d(v_i, v_j) \geq 3$;
(iii) $L \cap V(C_G) = \{v_i, v_{i+2}\}$ and $L \cap V(T_{i+3})$ is a maximum independent set in $T_{i+3}$.

Proof Let $E(L)$ be a fan associated with $L$ satisfying the properties in Lemma 6. Since there are $\alpha(G)$ components in $G - E(L)$ in all three cases, by Theorem 21, it suffices to show that $\{V(H) \mid H \text{ is a component in } G - E(L)\}$ is a core cluster of $G$. Note that, for $v_i \in L$, $v_i v_{i+1}$ is the unique incident edge of $v_i$ in $E(L)$. In cases (i) and (ii), for each component $H$ in $G - E(L)$, if we choose $e_u$ to be an edge $uv$ in $E(L)$ and let $E_H^+ = \{e_u \mid u \in V(H)\}$, then the extended graph $H + E_H^+$ induces no cycles in $G$. $G$ is realizable by Theorem 16, Lemma 7 and Theorem 9. Next, let us consider case (iii). In this case, the $v_i$’s, $i = 1, 2, \cdots, i, i + 3, i + 4, \cdots, k$, are in the same component, say $H$. It suffices to show that there exists an extended graph of $H$ whose vertex set induces no cycles in $G$. Since $L \cap V(T_{i+3})$ is a maximum independent set in $T_{i+3}$, $T_{i+3}$ is nontrivial and we know from Lemma 6 that $v_{i+3}$ is on some edge $v_{i+3}w$ in $E(L)$ where $w \in V(T_{i+3})$. Now, we let $e_{v_{i+3}}$ be the edge $v_{i+3}w$. For $u \in V(H) \setminus \{v_{i+3}\}$, we let $e_u$ be any incident edge of $u$ in $E(L)$. Since $v_{i+3} \notin V(H + E^+)$, where $E^+ = \{e_u \mid u \in V(H)\}$, the extended graph $H + E^+$ induces no cycles in $G$ and therefore $G$ is realizable by Theorem 16, Lemma 7 and Corollary 10.

The operation of replacing the edge $e$ of $G$ with a path of length $k + 1$ is called a $k$-subdivision of the edge $e$. This operation will be used in the proof of the next result.

Theorem 24 If $G$ is a unicycle graph of order $n$ with girth($G$) $\geq 5$, then

$$2 - \frac{\alpha(G) + 1}{n} \leq AR(G) \leq 2 - \frac{\alpha(G)}{n}.$$ 

Proof Let $C_G = (v_1, v_2, \cdots, v_k)$ be the unique cycle of $G$. We consider the case where $G$ is an odd unicycle graph with at least one Type I subtree or an even
unicycle graph first. We choose a fan \( E_L \) obtained from Lemma 6 associated with a suitable maximum independent set \( L \). By Proposition 3 and Theorem 23, it suffices to only consider the following two cases.

Case (i): \(| L \cap V(G) \cap C| = 1 \). We may assume that \( L \cap V(G) = \{v_k\} \). Let \( \{H_1, H_2, \ldots, H_{\alpha(G)}\} \) be the components in \( G - E_L \) and \( v_1 \in H_1 \), then \( V(C) \subseteq V(H_1) \). Denote as \( V' \) the set \( V(H_1) \setminus \{u \in V(T_j) \mid j = \lceil \frac{k}{2} \rceil + 1, \ldots, k \} \) and as \( V'' \) the set \( V(H_1) \setminus V' \). Let us choose the designated outside neighbor \( \bar{u} \) of any vertex \( u \) as the other endvertex of \( u \bar{u} \in E_L \). Observe that although \( v_k \) and \( v_1 \) are designated outside neighbors of each other, \( v_{\lceil \frac{k}{2} \rceil} \) and \( v_{\lceil \frac{k}{2} \rceil} + 1 \) have designated outside neighbors in \( T_{\lceil \frac{k}{2} \rceil} \) and \( T_{\lceil \frac{k}{2} \rceil + 1} \) respectively by Lemma 6.

The set of all designated outside neighbors of the vertices in \( V" \) is independent in \( G \), and so is the one for \( V" \). Therefore, \( \{V', V'', V(H_2), \ldots, V(H_{\alpha(G)})\} \) is a core cluster of \( G \) and \( \varepsilon(G) \leq \alpha(G) + 1 \). The result follows by Theorem 21.

Case (ii): \(| L \cap V(C) \cap C| = 2 \). By Theorem 23, we may assume that \( L \cap V(C) = \{v_k, v_2\} \). Let \( \{H_1, H_2, \ldots, H_{\alpha(G)}\} \) be the components in \( G - E_L \) and \( V_3 \subseteq H_1 \), then \( \{v_i \mid i = 3, \ldots, k\} \subseteq V(H_1) \). Denote as \( V' \) the set \( V(H_1) \setminus \{u \in V(T_j) \mid j = \lceil \frac{k}{2} \rceil + 1, \ldots, k \} \) and as \( V'' \) the set \( V(H_1) \setminus V' \). By choosing the designated outside neighbor \( \bar{u} \) of any vertex \( u \) as the other endvertex of \( u \bar{u} \in E_L \), we have that \( \{V', V'', V(H_2), \ldots, V(H_{\alpha(G)})\} \) is a core cluster of \( G \) and \( \varepsilon(G) \leq \alpha(G) + 1 \).

Next, we consider odd unicycle graphs without Type I subtrees. Let us 5-subdivide the edge \( v_3v_5 \) by replacing the edge with the path \( v_3 - v_4 - w_3 - w_4 - w_5 - v_5 \) and denote the graph after subdivision as \( G' \). Then \( G' \) is a realizable even unicycle graph by Corollary 20 and there exists an optimal core cluster \( \varepsilon(G') \) of size \( \alpha(G') \). Let \( L' \) be a suitable maximum independent set in \( G' \). Since \( G' \) does not contain Type I subtrees, we may assume that \( L' \cap V(C) = \{v_1, v_3, \ldots, v_k, w_2, w_4\} \). Let \( C_{G'} = (v_1, v_3, \ldots, v_k, w_2, w_4) \) where \( v_1, v_2, \ldots, v_k \) and \( w_3, w_4 \) are designated outside neighbors of each other, \( \varepsilon(G') = \{V'_3\setminus\{v_5\}, V'_3\setminus\{w_5\}\} \) is a core cluster of \( G \) because \( v_1 \) and \( v_k \) lie in different cores. Hence \( \varepsilon(G) \leq \alpha(G') - 2 \). Therefore, we have \( \varepsilon(G) + 2 \leq \alpha(G') \leq \alpha(G) + 3 \), which implies that \( \varepsilon(G) \leq \alpha(G) + 1 \).

The result follows.

Next, we investigate this problem for unicycle graphs with smaller girth.

**Theorem 25** Let \( G \) be a unicycle graph of order \( n \) with \( girth(G) = 4 \) and \( \theta(G) = \sum_{i=1}^{4} \theta(T_i) \), then \( AR(G) = 2 - \frac{\theta(G)}{n} \) if \( G \) has at least two Type II subtrees, and \( 2 - \frac{\theta(G)}{n} \leq AR(G) \leq 2 - \frac{\theta(G)+1}{n} \) otherwise.

**Proof** Let \( C_G = \{v_1, v_2, v_3, v_4\} \) be the unique cycle of \( G \). First, we partition \( \{1, 2, 3, 4\} \) into two parts \( \{x_1, x_2^2\} \) and \( \{x_2, x_2^3\} \) such that \( v_2 \) is a neighbor of \( v_{x_1} \) and if \( G \) has at least two Type I subtrees, then each \( \{T_{x_i}, T_{x_i}^r\}, i = 1, 2 \), must contain at least one of them. Next, among the part \( \{x_2, x_2^3\} \), if there is any nontrivial Type II subtree in \( \{T_{x_i}, T_{x_i}^r\} \), then we let \( T_{x_i} \), be a nontrivial
Now, consider the set \( V \) trivial, \( \{ v \} \). Assigning the designated outside neighbor of \( \cdots \) of the vertices in \( \{ T^i_{x1}, T^i_{x2} \} \). Therefore, there are several possible situations for the types of the subtrees \( (T^i_{x1}, T^i_{x2}) \):

(S1) (nontrivial Type II, trivial Type II);
(S2) (nontrivial Type II, nontrivial Type II or Type I);
(S3) (Type I, trivial Type II);
(S4) (trivial Type II, trivial Type II);
(S5) (Type I, Type I).

Now, we construct \( \alpha(T^i_{x1}) + \alpha(T^i_{x2}) \) cores to cover the vertices in \( V(T^i_{x1}) \cup V(T^i_{x2}) \) in situations (S1) to (S4), while in situations (S5) we construct \( \alpha(T^i_{x1}) + \alpha(T^i_{x2}) + 1 \) cores to cover the vertices in \( V(T^i_{x1}) \cup V(T^i_{x2}) \) in the following way. If \( T^i \) is nontrivial, we let \( H_{i} \) be an optimal star decomposition of \( T^i \) with vertex-number sum \( n_{H_{i}} \) and \( \mathcal{V}_{i} = \{ V_{i,1}, V_{i,2}, \cdots, V_{i,\alpha(T^i_2)+1} \} \) be an optimal core cluster of \( T^i \) with \( v_{i} \in V_{i,1}, \) then \( n_{H_{i}} = 2V(T^i_1) - (\alpha(T^i_1) + 1) \). In situation (S1), we consider an optimal core cluster \( \mathcal{V}_{i}^{\prime} \) of \( T^i_{x1} \) constructed from Lemma 4, then \( |\mathcal{V}_{i}^{\prime}| = \alpha(T^i_1) + 1 = \alpha(T^i_1) + \alpha(T^i_{x2}) \) because \( T^i_{x2} \) is trivial. The cores in \( \mathcal{V}_{i}^{\prime} \) cover the vertices in \( V(T^i_{x1}) \cup V(T^i_{x2}) \). In situation (S2), we may assume that the optimal core cluster of \( T^i_{x1} \) is \( \mathcal{V}_{i}^{\prime} = \{ V_{i,1}^{\prime}, V_{i,2}^{\prime}, \cdots, V_{i,\alpha(T^i_2)+1}^{\prime} \} \) in which \( V_{i,1}^{\prime} = \{ v_{x2}^{\prime} \} \) and \( v_{x2} \in V_{i,2}^{\prime} \).

Consider the set \( V_{x2}^{\prime} \cup V_{x2}^{\prime} \) where \( V_{x2}^{\prime} \) is the core containing \( v_{x2} \) in the optimal core cluster \( \mathcal{V}_{i}^{\prime} \) of \( T^i_{x2} \). If we keep the original designated outside neighbor for each vertex in \( V_{x2}^{\prime} \cup V_{x2}^{\prime} \) except \( v_{x2} \), and let the one for \( v_{x2} \) be the neighbor of it in \( C_{G} \) other than \( v_{x2}^{\prime} \), then \( V_{x2}^{\prime} \cup V_{x2}^{\prime} \) is a core of \( G \). Therefore, \( V_{x2}^{\prime} \cup V_{x2}^{\prime} \cup V_{x2}^{\prime} \), \( V_{x2}^{\prime} \cup V_{x2}^{\prime} \cup V_{x2}^{\prime} \), \( V_{x2}^{\prime} \cup V_{x2}^{\prime} \cup V_{x2}^{\prime} \) cores which cover the vertices in \( V(T^i_{x1}) \cup V(T^i_{x2}) \).

In situation (S3), \( T^i_{x1} + v_{x2} \) has an optimal core cluster \( \mathcal{V}_{x1}^{\prime} = \{ V_{x1,1}^{\prime}, V_{x1,2}^{\prime}, \cdots, V_{x1,\alpha(T^i_2)+2}^{\prime} \} \) constructed from Lemma 3 which satisfies that \( V_{x1,1}^{\prime} = \{ v_{x2} \}, v_{x2} \in V_{x1,2}^{\prime} \) and the designated outside neighbor of \( v_{x2} \) is not \( v_{x2}^{\prime} \). By assigning the designated outside neighbor of \( v_{x2}^{\prime} \) as the neighbor of it in \( C_{G} \) other than \( v_{x2}^{\prime} \), the set \( V_{x2}^{\prime} \cup V_{x2}^{\prime} \) is a core of \( G \). Since \( T^i_{x2} \) is trivial, the sets \( V_{x2}^{\prime}, V_{x2}^{\prime} \cup V_{x2}^{\prime}, \cdots, V_{x2}^{\prime}, \alpha(T^i_2)+2 \) are \( \alpha(T^i_1) + \alpha(T^i_{x2}) \) cores which cover the vertices in \( V(T^i_{x1}) \cup V(T^i_{x2}) \). In situation (S4), since \( T^i_{x1} \) and \( T^i_{x2} \) are both trivial, \( \{ v_{x2} \} \) and \( \{ v_{x2}^{\prime} \} \) are cores of \( G \). We have \( \alpha(T^i_{x1}) + \alpha(T^i_{x2}) \) cores covering the vertices in \( V(T^i_{x1}) \cup V(T^i_{x2}) \).

In situation (S5), let \( \mathcal{V}_{x2}^{\prime} \) be an optimal core cluster of \( T^i_{x1} + v_{x2} \) constructed from Lemma 3 with \( V_{x2,1}^{\prime} = \{ v_{x2} \} \), \( v_{x2} \in V_{x2,2}^{\prime} \) and the designated outside neighbor of \( v_{x2} \) is not \( v_{x2}^{\prime} \). By assigning the designated outside neighbor of \( v_{x2}^{\prime} \) as the one it has for the core \( V_{x2,1}^{\prime} \) in the optimal core cluster \( \mathcal{V}_{x2}^{\prime} \) of \( T^i_{x2} \) and keeping the original ones for the other vertices, \( V_{x2,1}^{\prime} \cup V_{x2,2}^{\prime} \) is a core of \( G \). We therefore have \( \alpha(T^i_{x1}) + \alpha(T^i_{x2}) + 1 \) cores \( V_{x2,1}^{\prime} \cup V_{x2,2}^{\prime}, V_{x2,3}^{\prime}, \cdots, V_{x2,\alpha(T^i_2)+2}, V_{x2,2}^{\prime}, \cdots, V_{x2,\alpha(T^i_2)+2} \) which cover the vertices in \( V(T^i_{x1}) \cup V(T^i_{x2}) \).
In the case that \( G \) has at least two Type II subtrees, only situation (S5) is impossible to occur because of the way we partition \( \{1, 2, 3, 4\} \) and \( \{x_2, x_3^*\} \) in the first place. Since we have constructed \( \alpha(T_{x_1}) + \alpha(T_{x_1}^*) \) cores covering the vertices in \( V(T_{x_1}) \cup V(T_{x_1}^*) \) in situations (S1) to (S4), we have a core cluster \( C \) of \( G \) which is of size \( \sum_{i=1}^{2} (\alpha(T_{x_i}) + \alpha(T_{x_i}^*)) = \sum_{i=1}^{4} \alpha(T_i) = \theta(G) \).

If \( G \) contains at least three Type I subtrees, then situation (S5) must occur. If all four are of Type I, then there exists a suitable maximum independent set \( L \) in \( G \) such that \( V(C_G) \cap L = \emptyset \) and \( \alpha(G) = \theta(G) \). The bound on \( AR(G) \) has been shown in Proposition 3. Now, we consider the case where \( G \) contains exactly three Type I subtrees. We may assume that \( (T_{x_1}, T_{x_1}^*) \) is in situation (S5) and \( (T_{x_2}, T_{x_2}^*) \) is not. In previous discussion, we have constructed \( \alpha(T_{x_1}) + \alpha(T_{x_1}^*) + 1 \) cores covering \( V(T_{x_1}) \cup V(T_{x_1}^*) \) and \( \alpha(T_{x_1}) + \alpha(T_{x_1}^*) \) cores covering \( V(T_{x_2}) \cup V(T_{x_2}) \). Therefore, we have a core cluster \( C \) of \( G \) which is of size \( \sum_{i=1}^{4} \alpha(T_i) + 1 = \theta(G) + 1 \).

On the other hand, we define a complete multipartite covering \( II \) of \( G \) as \( II = \{C_G\} \cup (\bigcup_{J \in \mathcal{J}} H_J) \) where \( J = \{i\mid T_i \) is nontrivial\}. Note that \( 2|V(T_i)| = (\alpha(T_i) + 1) = 0 \) holds for any trivial subtree \( T_i \). The vertex-number sum of \( II \) can be evaluated as follows.

\[
n_{II} = 4 + \sum_{i \in J} n_{II_i}
= 4 + \sum_{i \in J} \left(2|V(T_i)| - (\alpha(T_i) + 1)\right) + \sum_{i \in \{1, 2, 3, 4\} \setminus J} \left(2|V(T_i)| - (\alpha(T_i) + 1)\right)
= 2 \sum_{i=1}^{4} |V(T_i)| - \sum_{i=1}^{4} \alpha(T_i)
= 2|V(G)| - \theta(G).
\]

If \( G \) has at least two Type II subtrees, then \( n_{II} = 2|V(G)| - |C| \) and therefore \( G \) is realizable and \( AR(G) = 2 - \frac{\theta(G)}{n} \). If \( G \) has exactly three Type I subtrees, then \( n_{II} = 2|V(G)| - |C| + 1 \) and we have the bound on \( AR(G) \).

Next, we consider unicycle graphs with a 3-cycle.

**Theorem 26** Let \( G \) be a unicycle graph of order \( n \) with girth \( G = 3 \) and \( \theta(G) = \sum_{i=1}^{3} \alpha(T_i) \), then \( AR(G) = 2 - \frac{\theta(G)}{n} \) if every \( T_i \) is of Type II, and \( 2 - \frac{\theta(G) + 1}{n} \leq AR(G) \leq 2 - \frac{\theta(G)}{n} \) otherwise.

**Proof** Let \( C_G = (v_1, v_2, v_3) \) be the unique cycle of \( G \). If \( T_i \) is nontrivial, we let \( II_i \) be an optimal star decomposition of \( T_i \) with vertex-number sum \( n_{II_i} \).

We define a complete multipartite covering \( II \) of \( G \) as \( II = \{C_G\} \cup (\bigcup_{i \in J} II_i) \),
where \( J = \{ i | T_i \text{ is nontrivial} \} \). It has the vertex-number sum

\[
\begin{align*}
  n_H &= 3 + \sum_{i \in J} n_{H_i} \\
  &= 3 + \sum_{i \in J} (2|V(T_i)| - (\alpha(T_i) + 1)) + \sum_{i \in \{1, 2, 3\} \setminus J} (2|V(T_i)| - (\alpha(T_i) + 1)) \\
  &= 2|V(G)| - \theta(G).
\end{align*}
\]

Let \( \mathcal{C}_i' = \{ V'_{i,1}, V'_{i,2}, \cdots, V'_{i,|\mathcal{C}_i'|} \} \) be the optimal core cluster of \( T_i + v_{i+1} \) constructed from Lemma 3 or Lemma 4 in which \( V'_{i,1} = \{ v_{i+1} \} \) and \( v_i \in V'_{i,2} \). If \( T_i \) is of Type I, then \( |\mathcal{C}_i'| = \alpha(T_i) + 2 \). If \( T_i \) is of Type II, then \( |\mathcal{C}_i'| = \alpha(T_i) + 1 \).

Now, let us consider the following possible cases. (i) If all three subtrees \( T_i \)'s are of Type II, then \( \mathcal{C}' = \cup_{i=1, 2, 3} (\mathcal{C}'_i \setminus V'_{i,1}) \) is a core cluster of \( G \) of size \( |\mathcal{C}'| = \sum_{i=1}^3 \alpha(T_i) = \theta(G) \) and \( G \) is realizable. (ii) If \( T_1 \) and \( T_2 \) are of Type II and \( T_3 \) is of Type I, then \( \mathcal{C}' = \cup_{i=1, 2, 3} (\mathcal{C}'_i \setminus V'_{i,1}) \) is a core cluster of \( G \) of size \( |\mathcal{C}'| = \sum_{i=1}^3 \alpha(T_i) + 1 = \theta(G) + 1 \) and the bound on \( AR(G) \) follows. (iii) If \( T_1 \) is of Type II and \( T_2 \) and \( T_3 \) are not, then \( V'_{2,2} \cup V'_{3,2} \) is a core of \( G \) and \( (\mathcal{C}'_1 \setminus \{ V'_{i,1} \}) \cup (\cup_{i=2, 3} (\mathcal{C}'_i \setminus \{ V'_{i,1}, V'_{i,2} \})) \cup \{ V'_{2,2} \cup V'_{3,2} \} \) is a core cluster of \( G \) of size \( |\mathcal{C}'| = \sum_{i=1}^3 \alpha(T_i) + 1 = \theta(G) + 1 \) and we have the bound on \( AR(G) \) as well. (iv) If all three subtrees are of Type I, then there exists a maximum independent set \( L \) in \( G \) such that \( L \cap C_G = \emptyset \) and \( \theta(G) = \alpha(G) \). The result follows by Proposition 3. Our proof is completed.

We summarize our results on the optimal average information ratio of unicycle graphs in the next theorem.

**Theorem 27** Let \( G \) be a unicycle graph of order \( n \) with a cycle \( C_G = (v_1, v_2, \cdots, v_k) \), then

\[
2 - \frac{\theta(G) + 1}{n} \leq AR(G) \leq 2 - \frac{\theta(G)}{n},
\]

where \( \theta(G) = \alpha(G) \) if girth\( (G) \geq 5 \) and \( \theta(G) = \sum_{i=1}^k \alpha(T_i) \) otherwise.

Furthermore, \( AR(G) = 2 - \frac{n(G)}{n} \) if one of the following conditions is satisfied.

(i) \( G \) is an odd unicycle graph with at least one Type I subtree or an even unicycle graph and \( G \) has a suitable maximum independent set \( L \) such that one of the following statements is true.

(a) \( |L \cap V(C_G)| \geq 3 \);
(b) \( L \cap V(C_G) = \{ v_i, v_j \} \) and \( d(v_i, v_j) \geq 3 \);
(c) \( L \cap V(C_G) = \{ v_i, v_{i+2} \} \) and \( L \cap V(T_i + 3) \) is a maximum independent set in \( T_{i+3} \).

(ii) \( k = 4 \) and \( G \) has at least two Type II subtrees;

(iii) \( k = 3 \) and every \( T_i \) is of Type II.
6 Conclusion

In this paper, we show the existence of optimal star decompositions and establish bounds on the optimal average information ratio for general graphs. We also propose conditions under which the value of the optimal average information ratio can be determined and subsequently some infinite classes of realizable bipartite graphs are introduced. This extends the main result in [19]. Furthermore, we determine the exact values of the optimal average information ratio of some infinite classes of unicycle graphs and also give a good bound on the ratio for all unicycle graphs. The difference between our upper bound and lower bound is very small especially when $n$ is large. It is worth noting that the exact value of the optimal average information ratio also serve as a lower bound on the unknown optimal information ratio for those graphs. Deriving meaningful lower bounds on the optimal (average) information ratio is in general much more difficult than deriving upper bounds for most graphs.

Based on the discussion in this paper, one can see that constructing a good core cluster of $G$ is essential in the problem of investigating the value of the optimal average information ratio of the graph $G$. Besides, for graphs of small girth, star coverings may not result in the least vertex-number sum. Some other complete multipartite subgraphs of $G$ must be used in a complete multipartite covering to achieve the best results. Both of these questions are challenging and also worth trying.

References