Note

Cyclically decomposing the complete graph into cycles

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Abstract

Let $m_1, m_2, \ldots, m_k$ be positive integers not less than 3 and let $n = \sum_{i=1}^{k} m_i$. Then, it is proved that the complete graph of order $2n + 1$ can be cyclically decomposed into $k(2n + 1)$ cycles such that, for each $i = 1, 2, \ldots, k$, the cycle of length $m_i$ occurs exactly $2n + 1$ times.

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1. Introduction

A Steiner triple system (STS) is an ordered pair $(V,B)$, where $V$ is a finite nonempty set of elements, and $B$ is a collection of 3-element subsets of $V$ called triples, such that each pair of distinct elements of $V$ occurs together in exactly one triple of $B$. The order of a Steiner triple system $(V,B)$ is the size of $V$, denoted by $|V|$.

From “graph decomposition” point of view, the existence of a Steiner triple system of order $v$ (STS($v$)) is equivalent to the existence of a decomposition of the complete graph $K_v$ of order $v$ into edge-disjoint triangles, denoted by $C_3$. It is not difficult to see the necessary condition for such a decomposition to exist is that $v \equiv 1$ or $3 \pmod{6}$. In fact, this condition was proved to be sufficient around 150 years ago by Kirkman \[4\]. An automorphism of a STS $(V,B)$ is a bijection $\phi: V \to V$ such that $\{x,y,z\} \in B$ if and only if $\{\phi(x),\phi(y),\phi(z)\} \in B$. A STS($v$) is cyclic if it has an automorphism that is a permutation consisting of a single cycle of length $v$, for example $(1,2,3,\ldots,v)$.

Cyclic Steiner triple systems do exist. In 1939, Peltesohn used the so-called difference method to settle the existence problem.

Theorem 1.1 (Peltesohn \[7\]). For all $v \equiv 1$ or $3 \pmod{6}$ except $v = 9$, there exists a cyclic STS($v$).

We move on to consider an analog of Steiner triple systems. An $m$-cycle system of order $v$ is a pair $(V,C)$, where $V = V(K_v)$ and $C$ is a collection of edge-disjoint $m$-cycles which partition the edge set of $K_v$. Let $\Pi$ be an automorphism group of the $m$-cycle system $(V,C)$ (i.e., a group of permutations on $v$ vertices leaving the collection $C$ of cycles invariant). If there is an automorphism $\phi \in \Pi$ of order $v$, then the $m$-cycle system $(V,C)$ is said to be cyclic. For an $m$-cycle system of $K_v$, the vertex set $V$ can be identified with $\mathbb{Z}_v$. It is easy to see the necessary conditions for such a decomposition are (i) $v$ is odd and (ii) $m \mid \binom{v}{3}$.

The study of “existence problem” of $m$-cycle systems started around 40 years ago. Recently, Alspach and Gavlas \[2\] and \v{S}ajna \[10\] proved that an $m$-cycle system exists as long as the above conditions are met. Thus, we have all the $m$-cycle systems for each $m \geq 3$.

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Similar to a cyclic Steiner triple system, we can also consider the existence of cyclic $m$-cycle systems. Actually, the earlier works on the existence of $m$-cycle systems give cyclic systems. The case when $m \equiv 0 \pmod{4}$ and $v \equiv 1 \pmod{2m}$ was obtained by Kotzig [5] and the case when $m \equiv 2 \pmod{4}$ and $v \equiv 1 \pmod{2m}$ was due to Rosa [8]. Furthermore, Rosa [9] proved that if $m$ is odd and $v \equiv 1 \pmod{2m}$ or if $m$ is an odd prime and $v \equiv m \pmod{2m}$, then $K_v$ can be decomposed into closed trails of length $m$. In the case when $m = 5$ or 7, Rosa proved that the closed trials were indeed cycles. Therefore, cyclic 5-cycle systems and cyclic 7-cycle systems are obtained. Recently, Buratti and Del Fra [3] proved that for each odd prime $p$, cyclic $p$-cycle system exists.

In 1981, the following problem was posed by Alspach [1].

Conjecture. Let $m_1, m_2, \ldots, m_k$ be positive integers not less than 3 such that $\sum_{i=1}^{k} m_i = \left(\binom{n}{2}\right)$ for odd $n$ (respectively, $\sum_{i=1}^{k} m_i = \left(\binom{n}{2}\right) - n/2$ for even $n$). Then $K_n$ (respectively, $K_n - F$) can be decomposed into cycles $C_1, C_2, \ldots, C_h$ such that the length of $C_i$ is $m_i$ for $i = 1, 2, \ldots, h$.

In this paper, we prove a special case of the conjecture, namely, we prove that if $m_1, m_2, \ldots, m_k$ are positive integers all at least 3, then the complete graph $K_{2n+1}$, where $n = \sum_{i=1}^{k} m_i$, has a cyclic decomposition into $k(2n+1)$ cycles such that for each $i = 1, 2, \ldots, k$, there are exactly $2n+1$ cycles of length $m_i$.

2. The main results

Throughout this paper, we shall use difference methods. The difference between two vertices $x$ and $y$ in the complete graph $K_n$ with $V(K_n) = \mathbb{Z}_n$ is $|x - y|$ or $n - |x - y|$, whichever is smaller. We will say that the edge $xy$ has difference $d$ if for each integer $x \in \mathbb{Z}_n$, $y \in \mathbb{Z}_n$, and $i \in \mathbb{Z}$, the set of differences possible in $V(K_n)$ is $\{x + i \cdot d \pmod{n} \mid x \in \mathbb{Z}_n\}$. Thus, the set of differences possible in $V(K_n)$ is $\{x + i \cdot d \mid x \in \mathbb{Z}_n\}$ whenever $n$ is even. For convenience, we shall use $G[D]$ to denote the subgraph of $G$ induced by the set of differences $D \subseteq \{1, 2, \ldots, \ell\}$. It is easy to check that $K_{2\ell+1}[i]$ is a disjoint union of cycles of length $(2\ell + 1)/(2\ell + 1, i)$, where $(2\ell + 1, i)$ denotes the greatest common divisor of $2\ell + 1$ and $i$. Clearly, if $(2\ell + 1, i) = 1$, then $K_{2\ell+1}[i]$ is a Hamiltonian cycle in $K_{2\ell+1}$. It should be mentioned that if cycles $C_i, \ldots, C_k$ are mutually distinct, then there exists a cyclic decomposition of $K_{2\ell+1}$ into cycles $C_i, \ldots, C_k$.

Notice that if $H$ is a subgraph of $K_{2\ell+1}$ such that each edge of $H$ has a distinct difference, then the graph $H + i$ obtained from $H$ by adding $i$ (mod $2\ell + 1$) to each vertex of $H$ is an isomorphic copy of $H$. The following results are given in [13] and will be used in the proof of Theorem 2.5.

Lemma 2.1 (Wu [13]). For positive integers $b$ and $s$, there exists a cycle $C$ of length $4s$ with difference set
\[
\{b, b+1, \ldots, b+4s-1\}
\]
in $K_n$ where $n$ is odd with $n \geq 2(b+4s-1) + 1$.

Lemma 2.2 (Wu [13]). Let $b$ and $s$ be positive integers.

1. There exists a cycle $C$ of length $4s+2$ with difference set
\[
\{b, b+1, \ldots, b+4s, b+4s+2\}
\]
in $K_n$ where $n$ is odd with $n \geq 2(b+4s+2) + 1$.

2. There exists a cycle $C$ of length $4s+2$ with difference set
\[
\{b, b+2, b+3, \ldots, b+4s+2\}
\]
in $K_n$ where $n$ is odd with $n \geq 2(b+4s+2) + 1$.

Note that one may use a consecutive block of integers to construct cycles of length congruent to 0 modulo 4 and/or an even number of cycles of length congruent to 2 modulo 4. For example, if $m_1 = 4s+2$ and $m_2 = 4t+2$, then applying (1) and (2) of Lemma 2.2 give cycles $C_1$ and $C_2$ of lengths $m_1$ and $m_2$ with difference sets $\{b, b+1, \ldots, b+4s, b+4s+2\}$ and $\{b+4s+1, b+4s+3, \ldots, b+4s+4t+3\}$, respectively, for any positive integer $b$.

For convenience, in the following lemmas, we use a typical odd cycle as in Fig. 1.

Lemma 2.3. For positive integers $a, b, c, r$, with $c = a + b$ and $r > c$, and a nonnegative integer $s$, there exists a cycle $C$ of length $4s+3$ with difference set $\{a, b, c, r+1, \ldots, r+4s-1\}$ in $K_n$ where $n$ is odd and $n \geq 2(r+4s-1) + 1$.
Theorem 2.5. See, for example, Fig. 2, where length 4

Thus a; a

v

+ c

In Lemma 2.3, if

Remark. The proof is divided into two cases.

Case 1: Either a or b is odd, say b.

The cycle C of length 4s + 3 is defined as the following:

An easy verification shows that the vertices of the cycle C are: for i = 0, 1, ..., s, v2i+1 = a + 2i, v′ 2i+1 = c + 2i, v2i = a − r − 2(i − 1), v′ 2i = c + r + 4s − 2i + 1, where all indices are taken modulo n, and the difference set is \{a, b, c, r, r + 1, ..., r + 4s − 1\}. Observe that since c = a + b and b is odd, it follows that a and c have opposite parity. Thus a, a + 2, ..., a + 2s and c, c + 2, ..., c + 2s have opposite parity and hence are distinct. Also, c + r + 4s − 1, c + r + 4s − 2, ..., c + r + 2s + 1 and a − r, a − r − 2, ..., a − r − 2s − 2 have opposite parity when considered modulo n and thus are distinct. Therefore, the vertices of C are distinct.

Case 2: Both a and b are even.

(i) r is even: Let e1 = a, e′ 1 = r + 4s − 2, e2 = r + 4s − 1, e′ 2 = c, e2s+2 = b, and for i = 3, 4, ..., 2s + 1, let ei = r + 4s − 2i + 3, e′ i = r + 2i − 6. Now, we define the vertices accordingly. Let v0 = 0, v1 = a, v′ 1 = r + 4s − 2 and for i = 1, 2, ..., s, v2i = a + r + 4s − 2i + 1, v′ 2i = c + r + 4s + 2i − 4, v2i+1 = a + 2i, and v′ 2i+1 = c + 4s − 2i.

(ii) r is odd: In this subcase, we let e1 = a, e′ 1 = r + 4s − 1, e2 = r + 4s − 2, e′ 2 = c, e2s+2 = b, and for i = 3, 4, ..., 2s + 1, e′ i = r + 4s − 2i + 2, e′ i = r + 2i − 5. Then according to the differences, we define the vertices for the cycle. Let v0 = 0, v1 = a, v′ 1 = r + 4s − 1, and for i = 1, 2, ..., s, v2i = a + r + 4s − 2i, v2i+1 = a + 2i, v′ 2i = c + r + 4s + 2i − 3, and v′ 2i+1 = c + 4s − 2i.

Remark. In Lemma 2.3, if c = a + b ± 2, then we can use a similar method mentioned above to construct a cycle of length 4s + 1 with difference set \{a, b, c, r, r + 1, ..., r + 4s − 4, r + 4s − 2\}. Notice that this construction will be used in Theorem 2.5. See, for example, Fig. 2, where a = 2, b = 3, c = 7, s = 3, and r = 9.
Next, we consider cycles of length \(4s+1\).

Lemma 2.4. For positive integers \(a, b, c, r, \) and \(r,\) with \(a = a + b + 1\) and \(r > c,\) and a nonnegative integer \(s,\) there exists a cycle \(C\) of length \(4s+1\) with difference set \(\{a, b, c, r, r + 1, \ldots, r + 4s - 3\}\) in \(K_n\) where \(n\) is odd and \(n \geq 2(r + 4s - 3) + 1.\)

Proof. Let us define the cycle \(C\) of length \(4s+1\) as

![Cycle Diagram](image)

Notice that the value of \(a + b\) is even (odd) and \(c\) is odd (even). By routine computation, it follows that the difference set is \(\{a, b, c, r, r + 1, \ldots, r + 4s - 3\},\) and the distinct vertices of \(C\) are: \(v_1 = a, v'_1 = b, v_2 = c + 2s - 3, v'_2 = c + r + 2s - 2,\) and for \(i = 1, 2, \ldots, s - 1, v_{2i} = c + 2i - 3, v'_{2i} = c + r + 4s - 1 - 2i, v_{2i+1} = c - r - 2i - 1,\) and \(v'_{2i+1} = c + 2i.\)

In order to prove the main theorem, we also need to use Skolem sequences, hooked Skolem sequences, and near Skolem sequences.

A Skolem sequence of order \(n\) is a sequence \((s_1, s_2, \ldots, s_{2n})\) such that for each \(j \in \{1, 2, \ldots, n\},\) there exists a unique \(i \in \{1, 2, \ldots, 2n\}\) such that \(s_i = s_{i+j} = j.\) It is proved by Skolem [12] that such a sequence exists if and only if \(n \equiv 0\) or 1 (mod 4).

A hooked Skolem sequence of order \(n\) is a sequence \((s_1, s_2, \ldots, s_{2n+1})\) such that \(s_{2n} = 0\) and for each \(j \in \{1, 2, \ldots, n\},\) there exists a unique \(i \in \{1, 2, \ldots, 2n - 1, 2n + 1\}\) such that \(s_i = s_{i+j} = j.\) These sequences are known to exist if and only if \(n \equiv 2, 3\) (mod 4) (see [6]).

An \(m\)-near Skolem sequence of order \(n\) \((m \leq n)\) is a sequence \((s_1, s_2, \ldots, s_{2n-m})\) of \(2n - 2\) integers which satisfies for every \(j \in \{1, 2, \ldots, n\} \setminus \{m\},\) there exists a unique \(i \in \{1, 2, \ldots, 2n - 2\}\) such that \(s_i = s_{i+j} = j.\) It is proved by Shalaby [11] that an \(m\)-near Skolem sequence of order \(n\) exists if and only if either (1) \(m\) is odd and \(n \equiv 0\) or 1 (mod 4) or (2) \(m\) is even and \(n \equiv 2\) or 3 (mod 4).

Remark that a Skolem sequence \((s_1, s_2, \ldots, s_{2n})\) of order \(n\) gives a partition of \(\{1, 2, \ldots, 3n\}\) into triples \(\{j, n + i, n + i + j\}\) for \(j = 1, 2, \ldots, n.\) Similarly, a hooked Skolem sequence gives a partition of \(\{1, 2, \ldots, 3n - 1, 3n + 1\}\) into triples \(\{j, n + i, n + i + j\}\) satisfying \(j + s_i = t_i\) \((1 \leq j \leq n)\) and an \(m\)-near Skolem sequence gives a partition of \(\{1, 2, \ldots, 3n - 2\} \setminus \{m\}\) into triples \(\{j, s_i, t_i\}\) satisfying \(j + s_i = t_i\) for each \(j \in \{1, 2, \ldots, n\} \setminus \{m\}\).

Now, we are ready for the proof of our main result.

Theorem 2.5. Let \(m_1, m_2, \ldots, m_k\) be positive integers not less than 3 such that \(n = \sum_{i=1}^{k} m_i.\) Then there exists a cyclic \((m_1, m_2, \ldots, m_k)\)-cycle system of order \(2n+1.\)

Proof. For convenience, let \(m_1, m_2, \ldots, m_k\) denote the integers which are congruent to 3 modulo 4, \(m_{i+1}, m_{i+2}, \ldots, m_2\) denote the integers which are congruent to 1 modulo 4, \(m_3, m_4, \ldots, m_3\) denote the integers which are congruent to 0 modulo 4, and thus \(m_1, m_2, \ldots, m_k\) will be the integers which are congruent to 2 modulo 4. It suffices to partition the set \(\{1, 2, \ldots, n\}\) into sets \(A_1, A_2, \ldots, A_k\) such that:

- \(|A_i| = m_i\) for each \(i\) with \(1 \leq i \leq k;\)
- each of the sets \(A_1, A_2, \ldots, A_k\) satisfies the conditions for the difference set given in Lemma 2.3;
- each of the sets \(A_{1+1}, A_{1+2}, \ldots, A_2\) satisfies the conditions for the difference set given in Lemma 2.4 or the remark after Lemma 2.3;
- each of the sets \(A_{2+1}, A_{2+2}, \ldots, A_1\) satisfies the conditions for the difference set given in Lemma 2.1; and
- each of the sets \(A_{1+1}, A_{1+2}, \ldots, A_k\) satisfies the conditions for the difference set given in either (1) or (2) in Lemma 2.2.
Case 1: Suppose that \( i_2 \equiv 0, 1 \pmod{4} \). Clearly, if \( i_2 = 0 \), then it is easy to define the sets \( A_1, A_2, \ldots, A_k \) by choosing the differences for the cycles of length congruent to 0 modulo 4 first followed by choosing the differences for those cycles of length congruent to 2 modulo 4 last, using \( n + 1 \) for \( n \) as necessary. In fact, after defining the sets \( A_1, A_2, \ldots, A_i \), if we left with a set \( \{b, b + 1, \ldots, n\} \) for some positive integer \( b \), then we can easily choose the differences for sets \( A_{i+1}, A_{i+2}, \ldots, A_k \).

Since \( i_2 \equiv 0, 1 \pmod{4} \), there exists a Skolem sequence of order \( i_2 \) such that the set \( \{1, 2, \ldots, 3i_2\} \) can be partitioned into triples \( \{i, s_i, t_i\} \) with \( i + s_i = t_i \) for \( i = 1, 2, \ldots, i_2 \). Suppose first that \( i_2 = i_1 \) so that there are no cycles of length congruent to 1 modulo 4. Then, the sets \( A_1, A_2, \ldots, A_i \) are defined as follows:

- \( 1, s_1, t_1 \in A_1 \)
- \( 2, s_2, t_2 \in A_2 \)
- \( \vdots \)
- \( i_1, s_{i_1}, t_{i_1} \in A_{i_1} \), and
- starting with \( 3i_1 + 1 \), assign the next \( m_1 - 3 \) consecutive integers to \( A_1 \), the next \( m_2 - 3 \) consecutive integers to \( A_2 \) and so on until assigning \( m_{i_1} - 3 \) consecutive integers to \( A_{i_1} \).

Observe that the differences left are \( \sum_{i=1}^{i_1} m_i + 1, \sum_{i=1}^{i_1} m_i + 2, \ldots, n \), and as remarked earlier, the sets \( A_{i_2+1}, A_{i_2+2}, \ldots, A_k \) are easily found.

Now suppose that \( i_2 > i_1 \). Suppose first that \( i_2 - i_1 \) is even. Define the sets \( A_1, A_2, \ldots, A_{i_1} \) as follows:

- \( 1, s_1, t_1 \in A_1 \)
- \( 2, s_2, t_2 \in A_2 \)
- \( \vdots \)
- \( i_1, s_{i_1}, t_{i_1} \in A_{i_1} \)
- \( i_1 + 2, s_{i_1+1}, t_{i_1+1} \in A_{i_1+1} \)
- \( i_1 + 1, s_{i_1+2}, t_{i_1+2} \in A_{i_1+2} \)
- \( \vdots \)
- \( i_2, s_{i_2-1}, t_{i_2-1} \in A_{i_2-1} \)
- \( i_2 - 1, s_{i_2}, t_{i_2} \in A_{i_2} \)
- starting with \( 3i_2 + 1 \), assign the next \( m_1 - 3 \) consecutive integers to \( A_1 \), the next \( m_2 - 3 \) consecutive integers to \( A_2 \) and so on until assigning \( m_{i_2} - 3 \) consecutive integers to \( A_{i_2} \).

The differences remaining are \( \sum_{i=1}^{i_1} m_i + 1, \sum_{i=1}^{i_1} m_i + 2, \ldots, n \) and the sets \( A_{i_2+1}, A_{i_2+2}, \ldots, A_k \) are easily found.

Now assume that \( i_2 - i_1 \) is odd. Then, by [11], there exists a 1-near Skolem sequence of order \( i_2 \) so that the set \( \{2, 3, \ldots, 3i_2 - 2\} \) can be partitioned into triples \( \{i, s_i, t_i\} \) with \( i + s_i = t_i \) for \( i = 2, \ldots, i_2 \). If there are no cycles of length congruent to 2 modulo 4, then we define \( A_1, A_2, \ldots, A_{i_2} \) as follows:

- \( 2, s_2, t_2 \in A_1 \)
- \( 3, s_3, t_3 \in A_2 \)
- \( \vdots \)
- \( i_1 + 1, s_{i_1+1}, t_{i_1+1} \in A_{i_1} \)
- \( i_1 + 3, s_{i_1+2}, t_{i_1+2} \in A_{i_1+1} \)
- \( i_1 + 2, s_{i_1+3}, t_{i_1+3} \in A_{i_1+2} \)
- \( \vdots \)
- \( i_2 - 1, s_{i_2}, t_{i_2} \in A_{i_2-1} \)
- \( 1, n - 1, n + 1 \in A_{i_2} \), and
- starting with \( 3i_2 - 1 \), assign the next \( m_1 - 3 \) consecutive integers to \( A_1 \), the next \( m_2 - 3 \) consecutive integers to \( A_2 \) and so on until assigning \( m_{i_2} - 3 \) consecutive integers to \( A_{i_2} \).
(Note that if \( i_1 + 1 = i_2 \), then \( 1, n - 1, n + 1 \in A_{i_2} \).) If there are cycles of length congruent to 2 modulo 4, then we define the sets \( A_1, A_2, \ldots, A_{i_0}, A_{i_0 + 1} \) as follows:

- \( 2, s_2, t_2 \in A_1, \)
- \( 3, s_3, t_3 \in A_2, \)
  
  \[ \vdots \]
  
  - \( i_1 + 1, s_{i_1 + 1}, t_{i_1 + 1} \in A_{i_1}, \)
  - \( i_1 + 3, s_{i_1 + 2}, t_{i_1 + 2} \in A_{i_1 + 1}, \)
  - \( i_1 + 2, s_{i_1 + 3}, t_{i_1 + 3} \in A_{i_1 + 2}, \)
  
  \[ \vdots \]
  
  - \( i_2 - 1, s_{i_2}, t_{i_2} \in A_{i_2 - 1}, \)
  - \( 1, s_{i_2}, 1 \in A_{i_2}, \)
  - \( A_{i_0 + 1} = \{3i_2, 3i_2 + 2, 3i_2 + 3, \ldots, 3i_2 + mi_3 + 1\}, \)
  - \( \text{starting with } 3i_2 + mi_3 + 1, \text{ assign the next } mi - 3 \text{ consecutive integers to } A_1, \text{ the next } mi - 3 \text{ consecutive integers to } A_2 \text{ and so on until assigning } mi - 3 \text{ consecutive integers to } A_{i_2}. \)

In either case, the differences remaining form a set of consecutive integers, and the sets \( A_{i_2 + 1}, A_{i_2 + 2}, \ldots, A_{i_3 - 1}, A_{i_3 + 1}, \ldots, A_k \) are easily found.

Now, we have completed the partition of \( \{1, 2, \ldots, n\} \) into the sets \( A_1, A_2, \ldots, A_k \). By Lemmas 2.1–2.4, we are able to obtain a cyclic cycle decomposition for \( K_{2n+1} \). This concludes the proof of the case when \( i_2 \equiv 0 \) or \( 1 \) (mod 4).

**Case 2: Suppose that \( i_2 \equiv 2, 3 \) (mod 4).** The proof can be obtained by using a procedure similar to those in Case 1, and so we omit the details. \( \square \)

For clearness, we present two examples to show the idea of partition.

**Example 1.** \( n = 51. \) \( (m_1, m_2, \ldots, m_{10}) = (3, 3, 7, 5, 9, 4, 4, 6, 6), i_2 = 5. \) \( A_1 = \{1, 6, 7\}, A_2 = \{2, 12, 14\}, A_3 = \{3, 8, 11, 16, 17, 18, 19\}, A_4 = \{5, 9, 13, 20, 21\}, A_5 = \{4, 10, 15, 22, 23, 24, 25, 26, 27\}, A_6 = \{28, 29, 30, 31\}, A_7 = \{32, 33, 34, 35\}, A_8 = \{36, 37, 38, 39\}, A_9 = \{40, 41, 42, 43, 44, 46\}, \) and \( A_{10} = \{45, 47, 48, 49, 50, 51\}. \)

**Example 2.** \( n = 51. \) \( (m_1, m_2, \ldots, m_{10}) = (3, 3, 7, 5, 9, 4, 4, 6, 6), i_2 = 7. \) \( A_1 = \{1, 8, 9\}, A_2 = \{3, 14, 17\}, A_3 = \{4, 11, 15\}, A_4 = \{5, 13, 18, 22, 23, 24, 25\}, A_5 = \{7, 10, 16, 26, 27\}, A_6 = \{6, 12, 19, 28, 29\}, A_7 = \{2, 20, 21, 30, 31, 32, 33, 34, 35\}, A_8 = \{36, 37, 38, 39\}, A_9 = \{40, 41, 42, 43, 44, 46\}, \) and \( A_{10} = \{45, 47, 48, 49, 50, 51\}. \)

With the theorem we proved, the following known result can be obtained easily.

**Corollary 2.6.** For each \( m \geq 3 \) there exists a cyclic \( m \)-cycle system of order \( v \equiv 1 \) (mod \( 2m \)).

**Proof.** Since \( v = 2km + 1 \) for some integer \( k \geq 1, \) by Theorem 2.5 where we choose \( n = mk, \) we conclude the proof. \( \square \)

**Remark.** By an independent effort, we can also obtain the consequence, that is, for each odd prime \( p, \) there exists a cyclic \( p \)-cycle system.

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**References**

[13] S.L. Wu, Even \((m_1, m_2, \ldots, m_r)\)-cycle systems of the complete graph, Ars Combin., to appear.