Three public problems

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The existence problem

for Skolem 1-factorizations

On Skolem 1-factorizations

A Skolem 1-factor over Z_{2n+1} is a 1-factor $\{(a_i, b_i): 1 \le i \le n\}$ of K_{2n} on the set $\{1, 2, \cdots, 2n\}$ satisfying the Skolem condition:

$$b_i - a_i \equiv i \mod 2n+1,\ 1 \le i \le n,$$

which is denoted by $SF(n)$. A Skolem 1-factorization over Z_{2n+1} is a family of $SF(n)$s, which exactly forms a partition of all 2-subsets from $\{1, 2, \cdots, 2n\}$. The configuration is also regarded as a large set of Skolem 1-factors, so it is denoted by $LSF(n)$. Obviously, $LSF(n)$ consists of $2n-1$ disjoint $SF(n)$s.

Define an array $A(n) = ((a_{i,j}, b_{i,j}))_1^{n,2n\text{-}1}$, where $(a_{i,j}, b_{i,j}) =$

$(j, i+j)$ when $1 \le j \le 2n-i$, or

$(j+1, i+j-2n)$ when $2n+1-i \le j \le 2n-1$

A cross of $A(n)$ is a set $\{(a_{i,j_i}, b_{i,j_i}) : 1 \le i \le n\}$ of n items of $A(n)$, such that $\{\{a_{i,j_i}, b_{i,j_i}\} : 1 \le i \le n\} = Z_{2n+1}^*$.

Define a symmetric square $L(n) = (l_{i,j})_1^{2n}$, where

$l_{i,j} = j - i$ when $i < j \le n+i$,

$l_{i,j} = 2n+1+i-j$ when $n+1+i \le j \le 2n$;

$l_{i,j} = l_{j,i}$ when $i > j$; or $l_{i,i}$ is empty for $1 \le j \le 2n$.

A symmetric transversal of $L(n)$ is a set $\{(i_k, j_k) : 1 \le k \le n\}$ such that $\{l_{i_k, j_k}, l_{j_k, i_k} : 1 \le k \le n\} = \{1, 1, 2, 2, \cdots, n, n\}$.

$SF(n)$ is a cross of $A(n)$;

$LSF(n)$ is a a cross decomposition of $A(n)$;

$SF(n)$ is a symmetric transversal of $L(n)$;

$LSF(n)$ is a symmetric transversal decomposition of $L(n)$;

$SF(n)$ is equivalent to a cyclic almost resolvable B[2,1;2n+1].

unique $LSF(4)$

(1, 2),(4, 6),(5, 8),(3, 7);

(7, 8),(3, 5),(1, 4),(2, 6);

(2, 3),(6, 8),(4, 7),(1, 5);

(6, 7),(1, 3),(2, 5),(4, 8);

(3, 4),(5, 7),(8, 2),(6, 1);

(5, 6),(2, 4),(7, 1),(8, 3);

(4, 5),(8, 1),(3, 6),(7, 2).

There exist no $LSF(2)$.

There exist no $LSF(3)$.

There exist no $LSF(5)$.

Four nonisomorphic *LSF* (6)

First

(1,2),(8,10),(3,6),(5,9),(7,12),(11,4);
(11,12),(6,8),(2,5),(3,7),(9,1),(4,10);
(2,3),(4,6),(9,12),(7,11),(5,10),(8,1);
(10,11),(3,5),(4,7),(8,12),(1,6),(9,2);
(3,4),(7,9),(8,11),(1,5),(10,2),(6,12);
(9,10),(5,7),(1,4),(12,3),(6,11),(2,8);
(4,5),(9,11),(12,2),(6,10),(3,8),(1,7);
(8,9),(1,3),(7,10),(2,6),(12,4),(5,11);
(5,6),(10,12),(11,1),(4,8),(2,7),(3,9);
(7,8),(2,4),(6,9),(10,1),(11,3),(12,5);
(6,7),(12,1),(5,8),(11,2),(4,9),(10,3).

Second

(1,2),(5,7),(6,9),(8,12),(11,3),(4,10);
(11,12),(3,5),(7,10),(4,8),(1,6),(9,2);
(2,3),(10,12),(4,7),(5,9),(6,11),(8,1);
(10,11),(7,9),(3,6),(1,5),(12,4),(2,8);
(3,4),(8,10),(2,5),(7,11),(9,1),(6,12);
(9,10),(6,8),(1,4),(12,3),(2,7),(5,11);
(4,5),(9,11),(12,2),(6,10),(3,8),(1,7);
(8,9),(4,6),(11,1),(3,7),(10,2),(12,5);
(5,6),(2,4),(8,11),(10,1),(7,12),(3,9);
(7,8),(1,3),(9,12),(2,6),(5,10),(11,4);
(6,7),(12,1),(5,8),(11,2),(4,9),(10,3).

Third

(1,2),(8,10),(3,6),(5,9),(7,12),(11,4);
(11,12),(6,8),(4,7),(1,5),(10,2),(3,9);
(2,3),(4,6),(7,10),(8,12),(9,1),(5,11);
(10,11),(1,3),(6,9),(4,8),(2,7),(12,5);
(3,4),(5,7),(9,12),(10,1),(6,11),(2,8);
(9,10),(3,5),(8,11),(2,6),(12,4),(1,7);
(4,5),(7,9),(12,2),(6,10),(11,3),(8,1);
(8,9),(2,4),(11,1),(3,7),(5,10),(6,12);
(5,6),(10,12),(1,4),(7,11),(3,8),(9,2);
(7,8),(9,11),(2,5),(12,3),(1,6),(4,10);
(6,7),(12,1),(5,8),(11,2),(4,9),(10,3).

Fourth

(1,2),(5,7),(8,11),(6,10),(12,4),(3,9);
(11,12),(3,5),(7,10),(4,8),(1,6),(9,2);
(2,3),(8,10),(6,9),(1,5),(7,12),(11,4);
(10,11),(7,9),(1,4),(2,6),(3,8),(12,5);
(3,4),(6,8),(9,12),(10,1),(2,7),(5,11);
(9,10),(4,6),(2,5),(8,12),(11,3),(1,7);
(4,5),(10,12),(3,6),(7,11),(9,1),(2,8);
(8,9),(2,4),(11,1),(3,7),(5,10),(6,12);
(5,6),(9,11),(4,7),(12,3),(10,2),(8,1);
(7,8),(1,3),(12,2),(5,9),(6,11),(4,10);
(6,7),(12,1),(5,8),(11,2),(4,9),(10,3).

LSF (7)

(1, 2),(3, 5),(8,11),(9,13),(7,12),(4,10),(14, 6),
(2, 3),(8,10),(9,12),(1, 5),(6,11),(13, 4),(7,14),
(3, 4),(12,14),(7,10),(5, 9),(11, 1),(2, 8),(6,13),
(4, 5),(10,12),(11,14),(2, 6),(3, 8),(7,13),(9, 1),
(5, 6),(11,13),(4, 7),(10,14),(12, 2),(3, 9),(1, 8),
(6, 7),(9,11),(2, 5),(14, 3),(8,13),(10, 1),(12, 4),
(8, 9),(4, 6),(10,13),(12, 1),(2, 7),(14, 5),(11, 3),
(9,10),(6, 8),(13, 1),(3, 7),(14, 4),(11, 2),(5,12),
(10,11),(2, 4),(3, 6),(8,12),(9,14),(1, 7),(13, 5),
(11,12),(5, 7),(1, 4),(6,10),(13, 3),(8,14),(2, 9),
(12,13),(7, 9),(14, 2),(4, 8),(1, 6),(5,11),(3,10),
(13,14),(1, 3),(5, 8),(7,11),(4, 9),(6,12),(10, 2),
(7, 8),(14, 1),(6, 9),(13, 2),(5,10),(12, 3),(4,11).

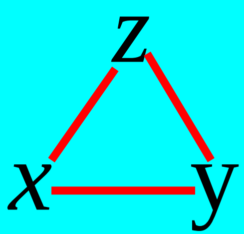
$LSF(8)$

(1, 2),(3, 5),(6, 9),(10,14),(11,16),(7,13),(8,15),(4,12),
(2, 3),(4, 6),(9,12),(11,15),(8,13),(16, 5),(7,14),(10, 1),
(3, 4),(5, 7),(10,13),(12,16),(9,14),(2, 8),(11, 1),(15, 6),
(4, 5),(9,11),(15, 1),(8,12),(2, 7),(10,16),(13, 3),(6,14),
(5, 6),(7, 9),(12,15),(16, 3),(13, 1),(8,14),(4,11),(2,10),
(6, 7),(1, 3),(11,14),(9,13),(5,10),(15, 4),(12, 2),(8,16),
(7, 8),(13,15),(1, 4),(6,10),(14, 2),(5,11),(9,16),(12, 3),
(8, 9),(16, 1),(7,10),(15, 2),(6,11),(14, 3),(5,12),(13, 4),
(15,16),(12,14),(8,11),(3, 7),(1, 6),(4,10),(2, 9),(5,13),
(14,15),(11,13),(5, 8),(2, 6),(4, 9),(12, 1),(3,10),(16, 7),
(13,14),(10,12),(4, 7),(1, 5),(3, 8),(9,15),(16, 6),(11, 2),
(12,13),(6, 8),(16, 2),(5, 9),(10,15),(1, 7),(14, 4),(3,11),
(11,12),(8,10),(2, 5),(14, 1),(16, 4),(3, 9),(6,13),(7,15),
(10,11),(14,16),(3, 6),(4, 8),(7,12),(13, 2),(15, 5),(1, 9),
(9,10),(2, 4),(13,16),(7,11),(15, 3),(6,12),(1, 8),(14, 5).

A line-chromatic problem

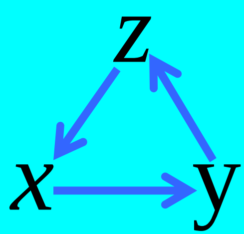
for a kind of 3-regular graphs

Six types of triples and the corresponding triple sysems



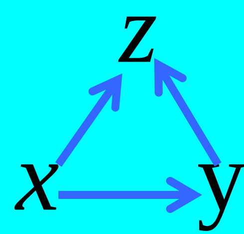
$\{x, y, z\}$

Steiner



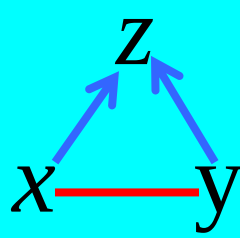
$\langle x, y, z \rangle$

Mendelsohn



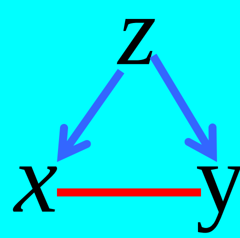
(x, y, z)

Directed



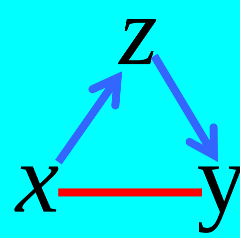
$[x, y, z]_1$

T_1



$[x, y, z]_2$

T_2



$[x, y, z]_3$

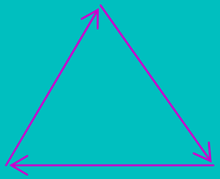
T_3

$STS(v)$	a partition of edges of	K_v	into	C_3
$MTS(v)$	a partition of edges of	DK_v	into	$\overrightarrow{C}_3$
$DTS(v)$	a partition of edges of	DK_v	into	TT_3
$HTS(v)$	a partition of edges of	DK_v	into	$\overrightarrow{C}_3$ or TT_3

triangle

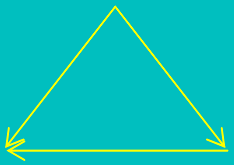
C_3

cyclic triangle



$\overrightarrow{C}_3$

transitive triangle

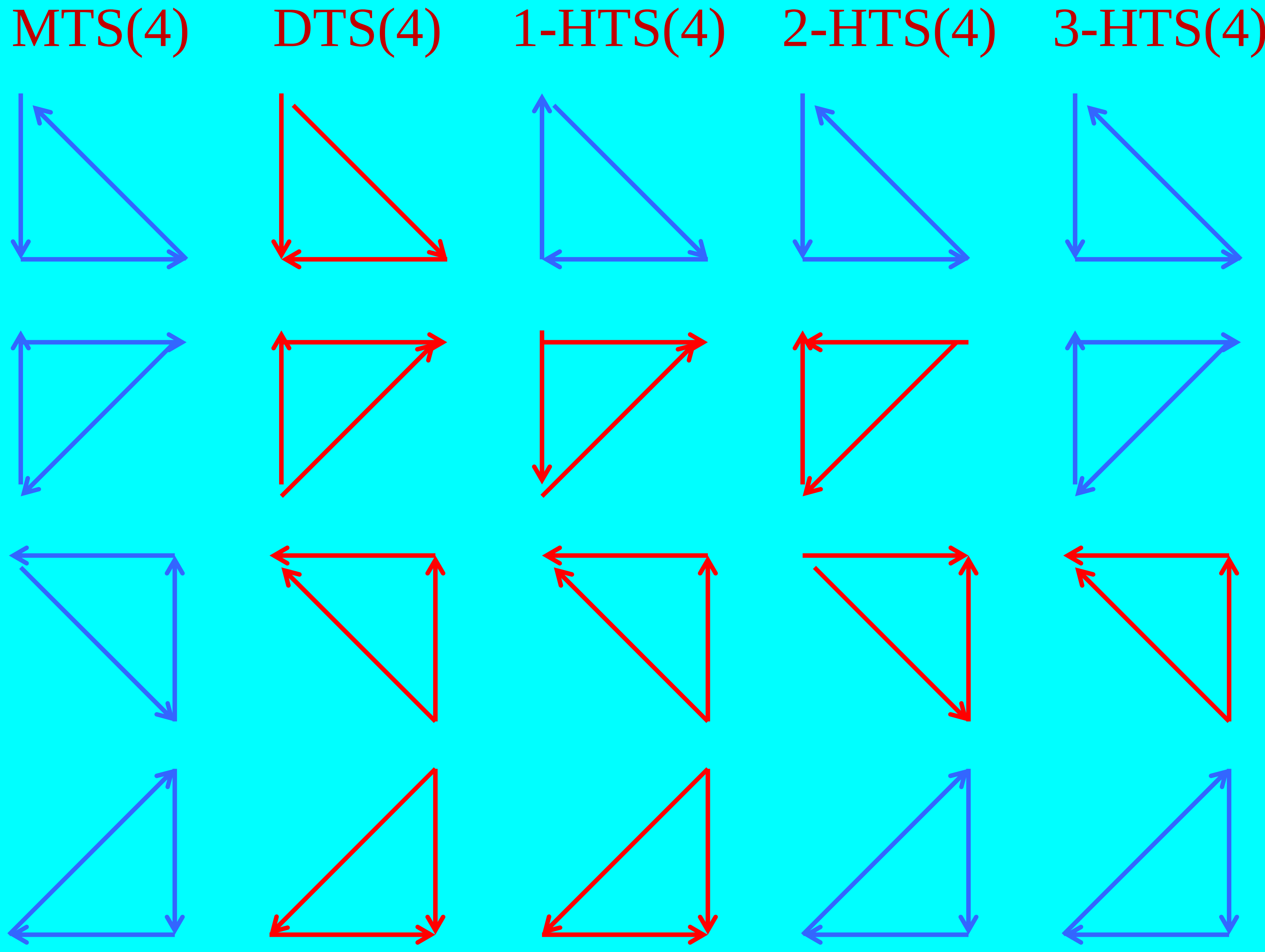


TT_3

$$\exists\, STS(v) \Leftrightarrow v \equiv 1,\ 3 \bmod 6$$

$$\exists\, MTS(v) \Leftrightarrow v \equiv 0, 1 \bmod 3,\ v \neq 6$$

$$\exists\, DTS(v)\ (HTS(v)) \Leftrightarrow v \equiv 0,\ 1 \bmod 3$$


MTS(4)
DTS(4)
1-HTS(4)
2-HTS(4)
3-HTS(4)

$MTS(v) \rightarrow$ three cyclic shifted $DTS(v)$

$MTS(v) = (X, \beta)$

Three cyclic shift of $B = \langle x, y, z \rangle \in \beta \rightarrow$

transitive triples $(x, y, z), (y, z, x), (z, x, y)$.

How to assign three cyclic shift of each $B \in \beta$ into three families D_1, D_2, D_3 respectively, such that each (X, D_i), $1 \le i \le 3$, is a $DTS(v)$?

If the problem is completely solved, then we may get the conclusion:

large set of some kind of $MTS(v) \rightarrow$

large set of same kind of $DTS(v)$

block-incidence graph of MTS

The block-incidence graph G of $MTS(v) = (X, \beta)$:

vertex set: $\beta - \overline{\beta}$, where $\overline{\beta}$ is all sub-$MTS(3)$ in β,

B and B' are joined if and only if $|B \cap B'| = 2$.

Graph G is 3-regular. If its line-chromatic number is just 3, then the above problem will be solved. Let the edges in G be partitioned three families G_1, G_2, G_3 with same colour respectively. For any edge $\{a,b\} \in G_j$ joining $\langle a,b,x\rangle$ and $\langle b,a,y\rangle$, let (b,x,a) and $(a,y,b) \in D_j$, $j = 1,2,3.$

For each sub-$MTS(3)$ on 3-set Y, let $g_j \in D_j$ where g_1, g_2 and g_3 form an $LDTS(3)$ on Y. Then, $(X, D_1), (X, D_2)$ and (X, D_3) are three disjoint $DTS(v)$.

Three nonisomorphic MTS(7)

013 124 235 346 450 561 602

310 421 532 643 054 165 206

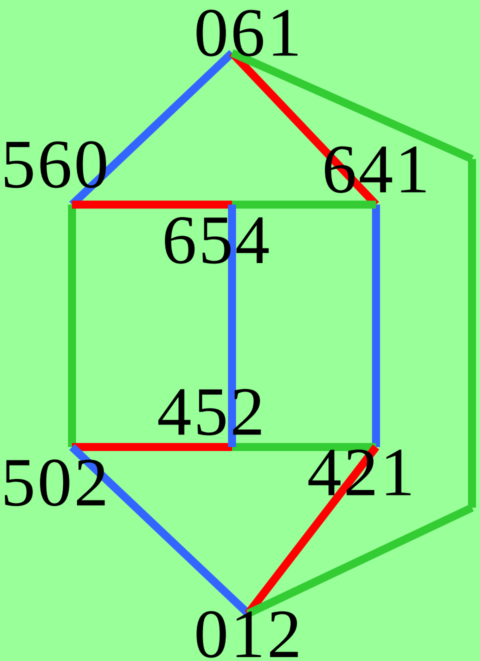
034 135 236 012 025 056 061

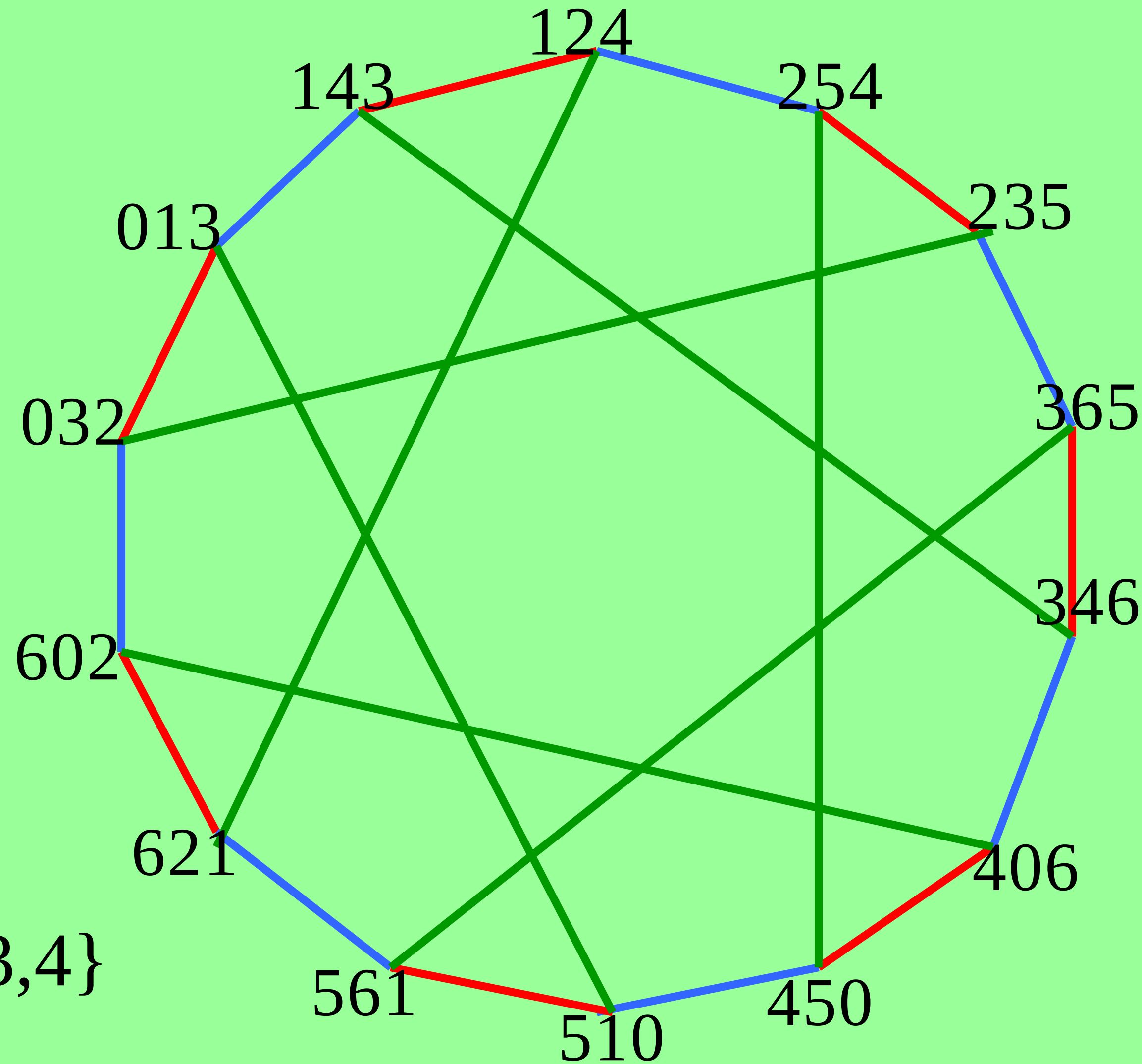
430 531 632 465 416 421 452

013 026 032 045 051 064 124

254 143 156 162 346 235 365

Block-incidence graph of MTS(7)



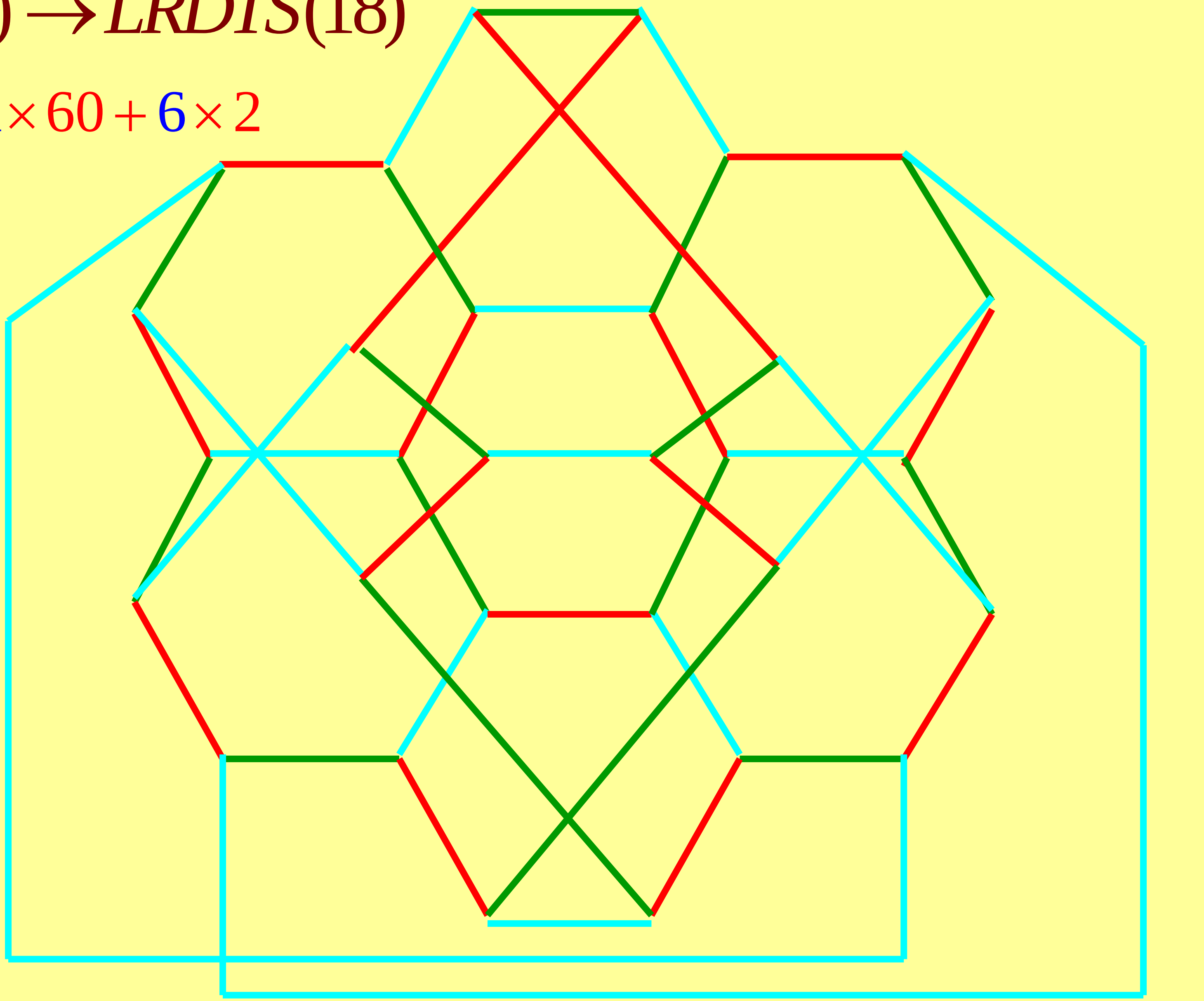


$sub-MTS(3)s$ on

$\{1,3,5\},\{2,3,6\},\{0,3,4\}$

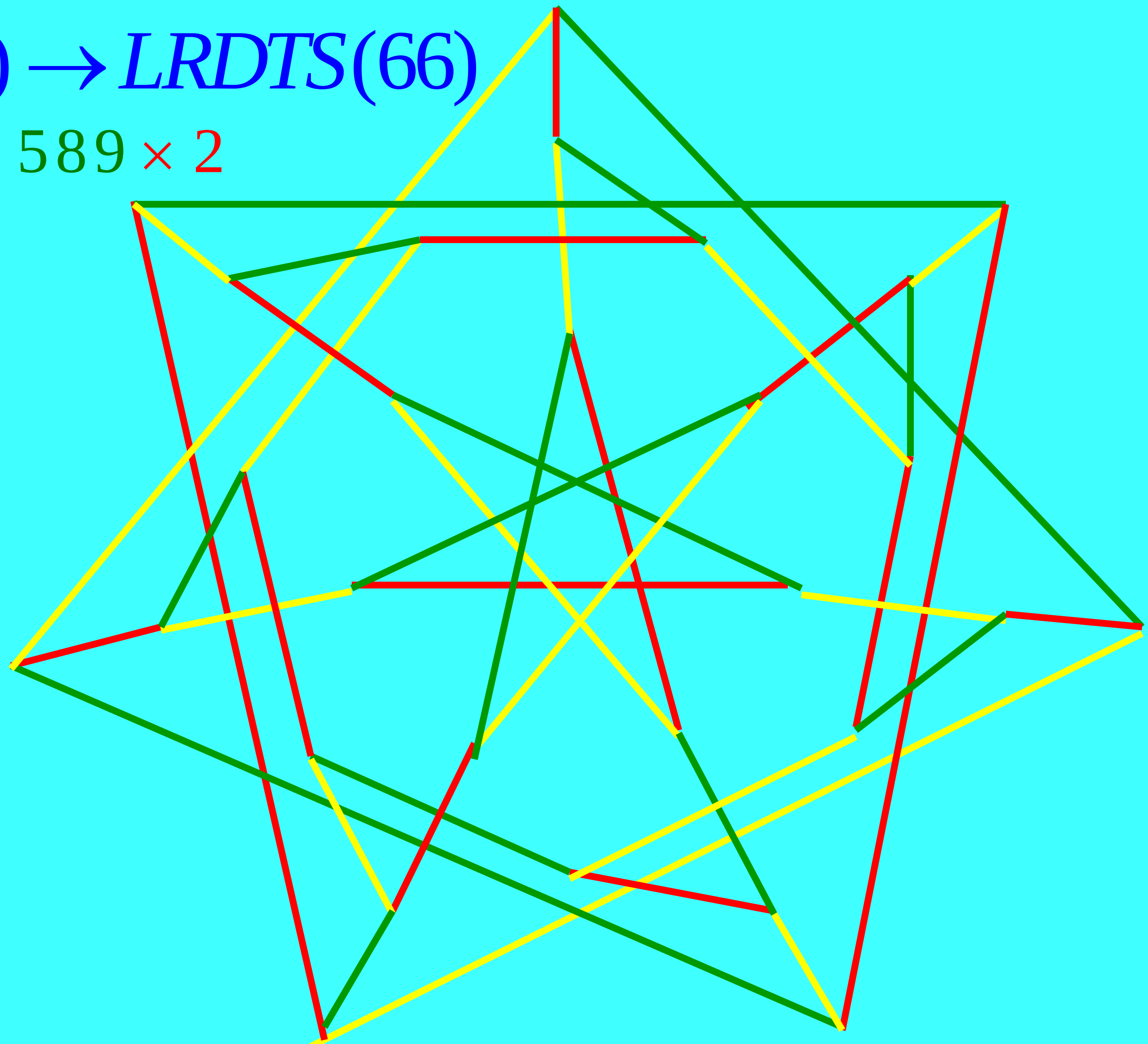
$LRMTS(18) \rightarrow LRDTS(18)$

$1 \times 30 + 1 \times 60 + 6 \times 2$



$LRMTS(66) \to LRDTS(66)$

$9 \times 28 + 589 \times 2$



Large set problem for Kirkman triple systems

大集问题的起源和背景

Kirkman's schoolgirl problem

(*T. P. Kirkman* 1847)

SUN			MON			TUE			WED			THU			FRI			SAT		
1	2	3	1	4	5	1	6	7	1	8	9	1	10	11	1	12	13	1	14	15
4	8	12	2	8	10	2	9	11	2	12	14	2	13	15	2	4	6	2	5	7
5	10	15	3	13	14	3	12	15	3	5	6	3	4	7	3	9	10	3	8	11
6	11	13	6	9	15	4	10	14	4	11	15	5	9	12	5	11	14	4	9	13
7	9	14	7	11	12	5	8	13	7	10	13	6	8	14	7	8	15	6	10	12

Thomas Penyngton Kirkman (英格兰教会的教区长)

<Lady's and Gentleman's Diary>

$$KTS(15) \quad \{a\} \cup (Z_7 \times Z_2) \ \bmod 7$$

{a,50,31},{01,41,51},{00,10,11},{20,40,61},{30,60,21}

$$LKTS(15) \quad \{a,b\} \cup Z_{13} \ \bmod 13$$

SUN			MON			TUE			WED			THU			FRI			SAT		
0	*a*	*b*	2	8	*b*	11	12	*b*	5	7	*b*	4	9	*b*	1	10	*b*	3	6	*b*
8	9	12	1	6	*a*	4	10	*a*	3	12	*a*	2	5	*a*	9	11	*a*	7	8	*a*
3	7	10	4	7	11	6	7	9	2	9	10	6	8	10	5	6	12	5	10	11
2	6	11	3	5	9	1	2	3	1	8	11	1	7	12	3	4	8	2	4	12
1	4	5	0	10	12	0	5	8	0	4	6	0	3	11	0	2	7	0	1	9

(1850 Sylvester, Cayley ——— 1974 Denniston)

KTS(9) ⟶ LKTS(9)

$$\{A, B\} \cup Z_7$$

124 ⋯ 124

214 ⋯ 356

AB ⋯ AB0

502 ⋯ 223

A46 ⋯ A2

B13 ⋯ B2

2	4	1
5	6	3
0	A	B

115 ⋯ 601

A1 ⋯ A23

B1 ⋯ B45

340 ⋯ 133

A51 ⋯ A3

B62 ⋯ B3

陆家羲与 Steiner三元系大集

1935年6月10日，陆家羲出生在上海市一个贫苦市民家。自幼聪慧过人，6岁入上海南浔路正德小学读书，学业一直保持优秀．1948年，正当他读初二时父亲患病辞世，勉强读到初中毕业即被迫辍学。1950年，到上海一个五金行学徒。1951年又只身到沈阳，考入东北电器工业管理局的统计训练班，后被分配到哈尔滨电机厂工作。家羲自学能力强，毅力过人，利用业余时间自修了全部高中课程，还基本上掌握了俄语，自学了英语和日语。

1957年夏，他阅读了我国老一辈数学家孙泽瀛编写的一本普及读物《数学方法趣引》，被其中的“Kirkman女生问题”和“Steiner系列问题” 吸引得如醉如痴。他跃跃欲试，萌生出一个顽强的念头：一定要攻克这两个世界著名数学难题——没想到，这竟决定了他日后一生的追求和道路。

1957年秋，他考入吉林师范大学(现东北师大)物理系，仅靠微薄的助学金开始了大学生活。 1961年大学毕业后，他被分配到包头钢铁学院任助教。 1962年该学院下马后又被调到包头市教育局，先后在教育局教研室，包头市八中、五中、二十四中及九中任物理教师，直到他去世。

从1957年到他去世前的26年中，他在自己心心思念的两个数学问题上投入了学习与工作之外的几乎所有业余时间：不分日夜、没有节假、理发周期一再增长、简单的饮食，… 乃至婚姻大事也一直被忽略到37岁之后。

从1961年到1983年，他共撰写了20余篇论文。除去他去世前后在美国JCT及国内数学学报上发表的之外，其余不是退稿就是石沉大海。

1. 30/12/1961, Construct orthogonal Latin squares using block designs;
2. 30/12/1961, A method to construct Kirkman and Steiner systems;
3. 14/3/1965, Constructing methods for BIBD and RBIBD;
4. 7/8/1965, A combinatorial method to construct orthogonal Latin squares;
5. 7/8/1965, On B[k,1;v] with prime power k;
6. 22/1/1966, On B[5, λ ;v];
7. 29/1/1966, A B[5,1;141];
8. 6/5/1978, On Kirkman problem I;
9. 6/5/1978, Combinatorial method to construct Kirkman systems;
10. 22/5/1978, On Kirkman problem II;
11. 2/7/1978, A new view for Kirkman problem;
12. 6/3/1979, On Kirkman quadruple systems;
13. 7/3/1979, A family of distinct Steiner triple systems;
14. 7/5/1979, Disjoint Steiner systems and Bays's conjecture;
15. 20/7/1979 The existence of resolvable balanced incomplete block designs;

16. 18/9/1981 On large sets of disjoint Steiner triple systems I, II;

17. 7/4/1982 On large sets of disjoint Steiner triple systems III;

18. 17/6/1982 On large sets of disjoint Steiner triple systems IV;

19. 4/3/1983 On large sets of disjoint Steiner triple systems V, VI.

1983年7月，他借钱来到大连出席全国首届组合数学会议, 并应邀在闭幕式上作了大会报告，受到与会中外学者的一致称赞。国际组合论界权威性刊物，美国的《Journal of Combinatorial Theory》 A辑分别在1983和1984年的两期上，以总共99个印刷页的惊人篇幅连载了他的6篇论文On large sets of disjoint Steiner triple systems （该文基本解决了长达140年之久的LSTS存在性问题，是国际上第一个完整的打击系列）。我国的《数学学报》也在1984年底发表了他解决柯克曼女生问题的重要论文 The existence of resolvable balanced incomplete block designs。华南师范大学开始商调他去任教，加拿大多伦多大学则邀他去合作研究。《Math. Review》也函请他担任评论员。1983年l0月在武汉召开的中国数学学会第四次全国代表大会更是破例邀请他作为唯一的中学代表在会上作报告。

1983年10月30日晚，他从武汉返回包头家中，兴奋地向妻子滔滔不绝地讲述着他这几个月来内心的感受：研究成果所受到的重视，国内外学术界给他的赞誉，自己进一步攻关的打算，… 但是积久的疲劳和长期潜伏的疾病，已远远超出他生理能够承受的极限。次日凌晨1时许，他突发“迷走性神经紊乱”，猝然与世长辞。终年才48岁。

1983年11月，包头市政府举行追悼会，对他颁发特别科学奖2000元，并在包头九中设立“陆家羲奖学金”。在他逝世1周年时，内蒙古政府召开“向优秀知识分子陆家羲学习表彰大会”，追授他特级教师称号及特别奖5000元。

1984年9月，国内十几位组合数学专家在呼和浩特市召开陆家羲学术工作评审会，对他的学术成果做出了如下的评价：

“……众所周知，1960年，Bose等证明了当t＞1时，关于4t＋2阶正交拉丁方的Euler猜想不成立；1961年Hanani给出并证明了k=3和4的(b,v,r,k, λ)设计存在的充要条件，这是区组设计理论中的两大举世闻名的成就，陆家羲关于大集的成果可以与上述两大成就相媲美，并将同它们一起载入组合数学的史册．”

1985年，在中国数学会理事长吴文俊等主持的首届刘微数学讨论班上专门安排了一个介绍陆家羲研究成果的学术报告。

1985—1986年，中国数学会并委托内蒙古数学会邀请组合设计界三位专家编辑出版了《陆家羲遗文集》,首次向国内外公布了对陆家羲在RB[3,1;v]和 RB[4,1, v] 存在性研究方面优先权的确认。

1988年8月，根据国内外学者的倡议，在安徽屯溪召开了以纪念陆家羲先生为主旨的“区组设计国际会议”。

1989年3月，陆家羲夫人代表他参加了在北京人民大会堂隆重举行的“1987年国家自然科学奖颁奖大会”，接受了我国自然科学界的最高荣誉—国家自然科学奖一等奖.

Known LKTS(v) and small orders ≤ 405

$LKTS(9)$ (Kirkman 1850) 9,

$LKTS(3^k)$ $k \geq 1$ (Sylvester 1893) 3, 27, 81, 243,

$LKTS(15)$ (Denniston 1974) 15,

$LKTS(3^k \cdot 11)$ $k \geq 1$ (Schreiber 1976, Wu 1990) 33, 99, 297,

$LKTS(3^k m)$ $m = 17, 35,$ $k \geq 1$ (Denniston 1979, L. Wu 1990)

51, 105, 153, 315,

$LKTS(3^k m)$ $m = 5, 25, 43,$ $k \geq 1$ (Denniston 1979)

45, 75, 129, 135, 225, 387, 405,

$LKTS(3^k \cdot 41)$ $k \geq 2$ (Y. Chang, G. Ge & L. Wu, 1999) 369,

$LKTS(201)$ (Y. Chang & G. Ge, 1999) 201,

$LKTS(3^k \cdot 91)$ $k \geq 1$ (G. Ge 2000) 273,

$LKTS(3^k m)$ $m = 7, 13,$ $k \geq 1$ (J. Zhou & Y. Chang, 2009)

21, 39, 63, 117, 189.

Tripling constructions for LKTS

$LKTS(v)$, $TKTS(v)$ $\rightarrow$ $LKTS(3v)$ (Denniston, 1979)

$LKTS(v)$, $TRISQ(v)$ $\rightarrow$ $LKTS(3v)$ (L. Zhu & S. Zhang, 2000)

$\exists\ TRISQ(v)\ \forall v \equiv 3 \bmod 6 \rightarrow$ $LKTS(v) \Rightarrow LKTS(3v)$

Product constructions for LKTS

$LKTS(v)$, $LR(u)$ $\rightarrow$ $LKTS(uv)$ $\leftarrow$ $LKTS(3v)$, $LR(u)$, $RPICS(v^3)$

(S. Zhang, L. Zhu, 2003)

(L. Ji & J. Lei, 2004)

The existence for LKTS(v)

$$v = 3^a 5^b m \prod_{i=1}^{u} (2 \cdot 7^{r_i} + 1) \prod_{j=1}^{w} (2 \cdot 13^{s_j} + 1)$$

$m \in \{1, 7, 11, 13, 17, 35, 43, 67, 91, 123\} \cup \{2^{2p+1} 25^q + 1 : p, q \geq 0\}$,

$a, r_i, s_j \geq 1,\ b, u, w \geq 0,\ \ a + u + w \geq 2$ for $b \geq 1, m \neq 1.$

Kirkman, Denniston, Schreiber, L.Wu, Y. Chang, G. Ge, L. Zhu, S. Zhang, J. Lei, L.Ji. … ***before 2005***

$$v = 3 \prod_{i=1}^{u} (2 q_i^{r_i} + 1) \prod_{j=1}^{w} (4^{s_j} - 1)$$

prime powers $q_i \equiv 7 \bmod 12,\ u + w \geq 1,\ r_i, s_j \geq 1.$

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unknown $LKTS(v) : v = 57, 69, 87, 93, 111, 123, 141, 147, 159, \cdots$