## Analysis -Spring, 2012

Goal: To discern how one approaches an analytical problem.

I. Integral calculus

1. (i) How does one show that, in Calculus, for a continuous function $f : \mathbb{R} \to \mathbb{R}$, the improper integral $\int_0^\infty f(x)dx$ converges if $\int_0^\infty |f(x)|dx$ converges?

Hint: $0 \leq A + |A| \leq 2|A|$.

(ii) Let $X$ be a Banach space, $f : \mathbb{R} \to X$, can one define Riemann integral $\int_a^b f(x)dx$, where $a, b \in \mathbb{R}$ ?

(iii) How would one consider convergence of improper integral $\int_0^\infty f(x)dx$ then?

(iv) Is it true that $\int_0^\infty \|f(x)\|dx$ converges $\Rightarrow \int_0^\infty f(x)dx$ converges ? If true, how would you justify it?

2. For a continuous function $f : \mathbb{R} \to \mathbb{R}$, let $z_n = \int_0^n f(x)dx$.

(i) Is it true that

$$\lim_{n\to\infty} z_n \text{ exists } \Rightarrow \int_0^\infty f(x)dx \text{ converges?}$$

(ii) Let $y_n = \int_0^n |f(x)|dx$. Is it true that

$$\lim_{n\to\infty} y_n \text{ exists } \Rightarrow \int_0^\infty f(x)dx \text{ converges?}$$

3. Discuss the validity of

$$g(\int_a^b f(x)dx) = \int_a^b g(f(x))dx,$$

and find conditions on $f$ and $g$ for such equality.

## II. Differential Calculus

1. Do you need a condition for

$$\frac{d}{dt}\int_0^\infty e^{-tx} f(x)dx = \int_0^\infty -xe^{-tx} f(x)dx?$$

2. Let $f(x)$, $0 \le x < \infty$, be a continuous and differentiable real valued function, $f(0) = 0$, and that $f'(x)$ is an increasing function of $x$ for $x \ge 0$. Prove that

$$g(x) = \begin{cases} \frac{f(x)}{x}, & \text{if } x > 0 \\ f'(0), & \text{if } x = 0 \end{cases}$$

is an increasing function of $x$.

## III. Uniform Continuity

1. (i) Give the definition of uniform continuity for function $f : \mathbb{R} \to \mathbb{R}$.

(ii) Give the definition of "uniformly differentiable" for function $f : \mathbb{R} \to \mathbb{R}$.

(iii) Show that the derivative of uniformly differentiable function is uniformly continuous.

2. Let $X$ be the space of all bounded uniformly continuous functions on $[0, \infty)$. Show that $X$ is a Banach space.

## IV. Mean Value Theorem

State the mean value theorem, for
(i) $f : \mathbb{R} \to \mathbb{R}$,
(ii) $f : \mathbb{R} \to \mathbb{R}^n$,
(iii) $f : \mathbb{R}^n \to \mathbb{R}^n$.

Give your comment on these versions of the mean value theorem.