交通大學應用數學系 101 學年度博士班入學考試

線性代數　　101/05/08

(1) (20%) Let V and W be vector spaces, and let $T : V \to W$ be linear. If V is finite-dimensional, prove that $\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

(2) Let $M_{2\times 2}$ be the vector space of all real 2×2 matrices. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}.$$

Define $L : M_{2\times 2} \to M_{2\times 2}$ by $L(X) = AXB$.

(a) (5%) Prove that L is a linear transformation.

(b) (15%) Compute the trace and the determinant of L.

(3) Let $A = \begin{pmatrix} 0.25 & 0.25 & 0.25 \\ 0.35 & 0.35 & 0.35 \\ 0.40 & 0.40 & 0.40 \end{pmatrix}$.

(a) (5%) Let λ be an eigenvalue of A. Prove that $|\lambda| \le 1$.

(b) (10%) Compute $\lim\limits_{n\to\infty} A^n$.

(c) (15%) Suppose $a_0 I + a_1 A + a_2 A^2 + A^3 = 0$. Find a_0, a_1 and a_2.

(4) (a) (20%) Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Define the adjoint of the operator T, which is symbolically denoted by T^*, by the unique linear operator on V satisfying $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$. Here $\langle\ ,\ \rangle$ is the inner product on V. Prove that the definition is well-defined.

(b) (10%) Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T(a_1, a_2) = (2ia_1 + 3a_2, a_1 - a_2)$. Find T^*.