101 學年度國立交通大學應用數學系博士班入學考試試題

數值分析

請詳列計算過程，僅有答案沒有計算過程，將不予以計分。

1. Consider the following iteration: $x_0 = 1$ and

$$x_{n+1} = \frac{1}{2}x_n + \frac{3}{2x_n}, \quad \text{for } n \geq 0.$$

   (a) Evaluate $\lim_{n\to\infty} x_n =?$ if it exists. (*10 points*)

   (b) Find the rate of convergence to the limit as $n \to \infty$. (*10 points*)

2. Given the following data $f(0) = 0$, $f(\frac{1}{3}) = \frac{1}{4}$ and $f(1) = 1$. Find a cubic spline that interpolates these data and satisfies the natural boundary conditions. (*10 points*)

3. (a) Determine constants a, b, c and α such that the following quadrature rule

$$\int_{-1}^{1} f(x)\,dx = af'(-\alpha) + bf'(\alpha) + cf(0),$$

   has the highest possible degree of precision. What is the highest degree of precision ? (*10 points*)

   (b) Use the quadrature rule in part (a) to approximate the following definite integral:

$$\int_0^{\pi} e^{3x} \sin 2x\,dx.$$

   You are allowed to use the constants a, b, c and α instead the exact values if you do not know how to solve part (a). Also, simplify your answer. (*10 points*)

4. (a) Let $A = (a_{ij})$ be a complex $n \times n$ matrix and define

$$R_i = \sum_{j\neq i}^{n} |a_{ij}|, \quad i = 1, 2, \cdots, n.$$

   Show that each eigenvalue of the matrix A is in at least one of the disks

$$\{z : |z - a_{ii}| \leq R_i\}.$$

   (*10 points*)

   (b) Use part (a), show that if an $n \times n$ matrix $B = (b_{ij})$ is strictly diagonally dominant, that is, for each $i = 1, \cdots, n$,

$$|b_{ii}| > \sum_{j\neq i}^{n} |b_{ij}|,$$

   then the matrix B is nonsingular. (*10 points*)

5. Consider the initial-value problem

$$\begin{cases} \frac{dy}{dt} = y - t^2 + 1, & 0 \le t \le 2, \\ y(t=0) = 0.5, \end{cases} \tag{1}$$

which has the exact solution $y(t) = (t+1)^2 - \frac{e^t}{2}$.
We define the mesh points $t_i = i\Delta t$ where $\Delta t = 2/N$, $i = 1, \cdots, N$ and N is the number of mesh points. The forward Euler method to solve the problem (1) is

$$\begin{cases} Y^{i+1} = Y^i + \Delta t(Y^i - t_i^2 + 1), & i = 0, 1, \cdots, N-1, \\ Y^0 = 0.5, \end{cases} \tag{2}$$

where Y^i is the approximation of $y(t)$ at $t = t_i$. Then, show that

$$\max_{0 \le i \le N} |y(t_i) - Y^i| \le 0.1(0.5e^2 - 2)(e^2 - 1).$$

(*20 points*)

6. Find the first two iteration of the Gauss-Seidel method for the following linear system, using $\mathbf{x}^{(0)} = \mathbf{0}$:

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5, \\ -x_1 + 3x_2 + x_3 &= -4, \\ 2x_1 + 2x_2 + 5x_3 &= 1. \end{aligned}$$

(*10 points*)