一百零二學年度國立交通大學應用數學系博士班入學考試試題

科目．分析

Please explain *all* your answers and indicate which theorems you are using!

1. (15 points) Let $\Omega \subset \mathbb{R}^2$ be *open*. Suppose the function $f : \Omega \longrightarrow \mathbb{R}$ has partial derivatives f_x and f_y everywhere in Ω, and these partial derivatives satisfy the inequalities
$$|f_x(x,y)| \le M, \quad |f_y(x,y)| \le M \qquad \forall\, (x,y) \in \Omega$$
where M is a constant independent of $(x,y) \in \Omega$. Is f continuous in Ω? Justify your answer! (Note that we do not assume that f differentiable in Ω.)

2. (15 points) Let $\mathbf{F} = \left(\arctan \dfrac{y}{x}, \dfrac{\ln(x^2+y^2)}{2}\right)$ be the smooth vector field defined in a neighborhood of the rectangle $R = [1,3] \times [-2,2]$. Compute $\int_{\partial R} \mathbf{F} \cdot d\mathbf{x}$, where R is oriented counterclockwisely.

3. Let $f : [0,1] \longrightarrow \mathbb{R}$. For each $n \in \mathbb{N}$, let $x_j = \dfrac{j}{n}$ for $j = 0,1,2,\ldots,n$. Define $f_n : [0,1] \longrightarrow \mathbb{R}$ by $f_n(x) = f(x_{j-1}) + \dfrac{f(x_j) - f(x_{j-1})}{n}(x - x_{j-1})$ for $x \in [x_{j-1}, x_j]$, for $n \in \mathbb{N}$ and $j = 0,1,2,\ldots,n$. (Notice that f_n is *piecewise linear.*)
   (a) Suppose that $f \in \mathscr{C}([0,1])$, that is f is continuous on $[0,1]$
      i. (5 points) Show that $f_n \to f$ pointwisely on $[0,1]$.
      ii. (5 points) Does $f_n \to f$ uniformly on $[0,1]$? Justify your answer.
   (b) (10 points) Suppose that f is Riemann integrable over $[0,1]$. What can be said about the pointwise and uniform convergence of f_n to f on $[0,1]$? Justify your answer(s)!

4. (15 points) Compute the following two iterated Lebesgue integrals
$$\int_{[0,1]} \int_{[1,\infty)} (e^{-xy} - 2e^{-2xy})\, dx\, dy \qquad \text{and} \qquad \int_{[1,\infty)} \int_{[0.1]} (e^{-xy} - 2e^{-2xy})\, dy\, dx,$$
and comment your the results in connection to the Fubini's theorem.

5. (15 points) Prove that $\lim_{n\to\infty} \int_{\mathbb{R}} (1 - e^{-x^2/n}) e^{-|x|} \sin^3 x\, dx = 0$.

6. (10 points) If $\|f\|_p < \infty$ for some $p \in (0,\infty)$. Prove or disprove that $\|f\|_p \to \|f\|_\infty$.

7. (10 points) Suppose that $f_k, f \in L^2$ and that $\int f_k g \to \int fg, \forall g \in L^2$. (i.e. $\{f_k\}$ converges to f weakly in L^2.) If $\|f_k\|_2 \to \|f\|_2$, show that $f_k \to f$ in L^2.