85学年度博士班入学考試考題

ANALYSIS

I (15 pts) **(A)** Give a definition of compactness of a subset of a topological space. Is the set $\{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$ compact in $\mathbb{R}$? Justify your answer.

(B) Let $E \subset \mathbb{R}^n$. Prove :

(a) Let $\{B_\alpha\}_{\alpha \in A}$ be a family of open balls such that $E \subset \cup_{\alpha \in A} B_\alpha$. Then there exists an at most countable set $A_0 \subset A$ such that $E \subset \cup_{\alpha \in A_0} B_\alpha$.

(b) There exists an at most countable subset of E whose closure contains E.

II (15 pts) **(A)** Let $\{a_n\}$ be a sequence of real number such that $a_n \downarrow 0$. Prove :

(a) The series $\sum a_n$ and $\sum 2^n a_{2^n}$ converge or diverge simultaneously.

(b) If $\sum a_n = \infty$, then $\sum \min(a_n, \frac{1}{n}) = \infty$. (**Hint:** Use (a).)

(B) Suppose $\lambda_n \to 0$. Does the series $\sum_n \lambda_n e^{-|x-n|}$ converge uniformly in $\mathbb{R}$?

III (15 pts) **(A)** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map so that its Jacobian never vanishes in $\mathbb{R}^2$. Can we draw the conclusion that f is one-to-one? Verify your conclusion.

(B) Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$v_n = 2v_{n-1} \int_0^1 (1-t^2)^{(n-1)/2}\, dt.$$

(C) Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

IV (10pts) Open intervals of length $1/3, 1/9$,etc. were removed successively from $[0, 1]$ in the construction of the Cantor set. Let $f(x) = n$ on the open intervals of length 3^{-n}, and let $f(x) = 0$ on the Cantor set. Prove that f is measurable, and find $\int_0^1 f(x)dx$.

V (15pts) Suppose that $E \subset \mathbb{R}^n$ with $m(E) < \infty$, and f is a strictly positive Lebesgue measurable function defined on E. Here $m(E)$ is the Lebesgue measure of the set E.

(A) Prove that if f is Lebesgue integrable in E, then

$$\int_E f^p(x) m(dx) \to m(E) \text{ as } p \to 0^+$$

(B) Prove that the following statements are equivalent:

(a) f is Lebesgue integrable in E.

(b) $\sum_{k\geq 1} m\{x \in E | f(x) \geq k\} < \infty$.

(c) $\sum_{k\geq 1} k m\{x \in E | k \leq f < k+1\} < \infty$.

VI (15pts) **(A)** Suppose that $a_j, b_j \geq 0$,for $j = 1, 2, \cdots, n$, and that $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Prove the Hölder 's inequality

$$\sum_{j=1}^n a_j b_j \leq (\sum_{j=1}^n a_j^p)^{\frac{1}{p}} (\sum_{j=1}^n b_j^q)^{\frac{1}{q}}.$$

(B) Suppose that $x_j > 0, j = 1, 2, \cdots, n$, and let

$$k(p) = [\frac{1}{n}(\sum_{j=1}^{n} x_j^p)]^{\frac{1}{p}} \text{ for } p > 0.$$

Prove that

(a) k is monotonically increasing.

(b) $\lim_{p\to\infty} k(p) = \max\{x_1, \cdots, x_n\}$.

VII (15 pts) (A) If $f_k \to f$ in $L^p, 1 \le p < \infty, g_k \to g$ pointwise, and $\|g_k\|_\infty \le M$ for all k, prove that $f_k g_k \to fg$ in L^p.

(B) Let $f, \{f_k\} \in L^p$. Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$. Conversely, if $f_k \to f$ *a.e.* and $\|f_k\|_p \to \|f\|_p, 1 \le p < \infty$, show that $\|f - f_k\|_p \to 0$.