NATIONAL CHIAO TUNG UNIVERSITY
DEPARTMENT OF APPLIED MATHEMATICS
PH.D. ANALYSIS ENTRANCE EXAM

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Instruction: There are two groups of problem sets: advanced calculus and real analysis. Choose at least three of them from each group. The problems are not arranged in accordance with their difficulty levels. Please read all problems first and do the ones that are easiest for you.

PART ONE: ADVANCED CALCULUS (40 POINTS)

Problem 1. *Let K be a subset of $\mathbb{R}^n$. Suppose every continuous real-valued function on K is bounded. Prove or disprove K is compact.*

Problem 2. *Suppose $f : \mathbb{R} \to \mathbb{R}$. Suppose $f'(x) > f(x)$ for all $x \in \mathbb{R}$ and $f(0) = 0$. Prove $f(x) > 0$ for all positive x.*

Problem 3. *Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$. Find differentiable real-valued functions f and g on Ω such that $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$ but there is no real-valued function h on Ω such that $f = \frac{\partial h}{\partial y}, g = \frac{\partial h}{\partial x}$. (Hint: Why would Green's theorem fail to apply?)*

Problem 4. *Let f be a real-valued continuous function defined on $\mathbb{R}$ satisfying the inequality*

$$f(x) \le \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy, \quad x \in \mathbb{R}, \quad h > 0.$$

Prove:

(1) *$f(x) \le \max(f(a), f(b))$, for all $x \in [a, b]$. Here $-\infty < a < b < \infty$.*
(2) *f is convex, i.e., the inequality*

$$f((1-t)a + tb) \le (1-t)f(a) + tf(b)$$

holds for $a < b$ and $0 \le t \le 1$.

PART TWO: REAL ANALYSIS (60 POINTS)

Problem 5. *For each of the following statements (1)–(5), give an example of real-valued functions f_n $(n = 1, 2, \cdots)$ or f defined on a measure space Ω in $\mathbb{R}$ such that the statement holds:*

(1) *(f_n) converges almost everywhere in Ω but not almost uniformly.*
(2) *(f_n) converges in the mean but (f_n) is not convergent almost everywhere in Ω.*
(3) *(f_n) is bounded in $L^2(\Omega)$ and converges almost everywhere in Ω, but (f_n) is not convergent in $L^2(\Omega)$.*
(4) *$f \in L^2(\Omega) \setminus L^1(\Omega)$.*
(5) *$f \in L^1(\Omega) \setminus L^2(\Omega)$.*

Problem 6. *Prove that a bounded closed set K of points $x = (x_1, x_2, \cdots)$ in ℓ^p is compact if for any $\epsilon > 0$ there is an n_0 such that $\sum_{n=n_0}^{\infty} |x_n|^p \leq \epsilon$ for all $x \in K$.*

Problem 7. *Let (X, μ) and (Y, ν) be measure spaces and consider the operator A defined by*

$$(Af)(x) = \int_Y K(x, y) f(y)\, d\nu(y)$$

from $L^r(Y, \nu)$ into $L^p(X, \mu)$, where $\frac{1}{p} + \frac{1}{r} = 1, 1 < p < \infty$. Prove that if $K(x, y) \in L^p(X \times Y, \mu \times \nu)$, then A is a bounded operator.

Problem 8. *Let Ω be a measure space, and $g(x, r)$ be defined on $\Omega \times \mathbb{R}$ such that, for every $r \in \mathbb{R}$, $g(x, r)$ is measurable with respect to $x \in \Omega$.*

(1) *Show that if $s(x)$ is a simple measurable function on Ω, then $g(x, s(x))$ is a measurable function in Ω.*

Assume that, for almost everywhere $x \in \Omega, g(x, r)$ is a continuous function with respect to $r \in \mathbb{R}$.

(2) *If $u(x)$ is a real-valued measurable function defined on Ω, then $g(x, u(x))$ is measurable on Ω.*

(3) *Show that if in addition $|g(x, r)| \leq h(x), h \in L^2(\Omega)$, then the map G, defined by $(G(u))(x) = g(x, u(x))$ for $u \in L^2(\Omega)$, is continuous from $L^2(\Omega)$ into $L^2(\Omega)$.*