86.6.4.

Instructions : There are 6 problems with various points as indicated. Answer as many questions as you can. Notations: K_n is the complete graph with n vertices, $\overline{K}_n$ is the completement of K_n, I is the identity matrix and J is the all-one matrix with suitable order.

I. Let G=<V,E> be a connected graph with n vertices named as 1, 2, ..., n. The distance $d(i,j)$ denotes the length of the shortest path between vertices i and j, for $i,j\in V$.

The separation number of a vertex $i\in V$ is defined as

$$s(i)=\max \{d(i,j) \mid j\in V\}.$$

The transmission number, denoted by $t(i)$, of a vertex $i\in V$, is the sum of the distance between i and all other vertices $j\in V$. That is, $t(i)=\Sigma_{j=1}^{n} d(i,j)$.

The centers of G are the vertices of G with the minimum separation number.

The medians of G are the vertices of G having the minimum transmission number.

(1) Let G be the graph as shown in Figure 1. Find the $d(i,j)$ for all vertices i and j. (3 points)

(2) Under the graph in Figure 1, use the breadth–first–search method to find the centers and medians of G. (3 points)

(3) For an arbitrary connected graph G, design an algorithm to find the centers and medians of G. (8 points)

(4) What is the time complexity of your algorithm ? Why ? (3 points)

(5) Use the graph in Figure 1 as an example to illustrate the work of your algorithm. (3 points)

II. Let G=<V,E> be a given connected graph with n vertices.

(1) What is a Hamiltonian path ? a cycle ? (3 points)

(2) Does there exist a bipartite graph G such that G has a Hamiltonian path but not a Hamiltonian cycle ? Why ? (3 points)

(3) Let G be a simple graph with $n\geq 3$ such that $\deg(u)+\deg(v)\geq n-1$ for every two distinct nonadjacent vertices u and v. Show that G has a Hamiltonian cycle. (4 points)

(4) For an arbitrary graph G, design a program to indicate whether there is a Hamiltonian cycle in G. (4 points)

(5) Use the graph G as shown in Figure 2 to illustrate the work of your program. (3 points)

(6) Modify your program to obtain the number of Hamiltonian cycle of a given graph G. (3 points)

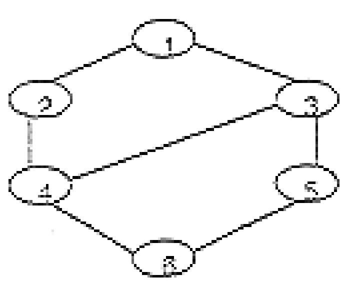

Figure 1

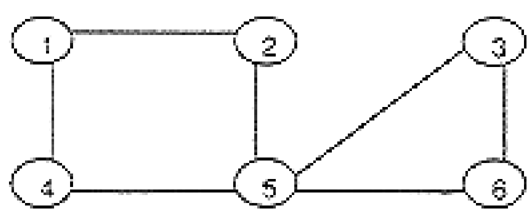

Figure 2

III. (30 points) For a finite simple graph $G = (V(G), E(G))$, we define a polynomial $f(G) = f(G; t, x, y)$ by the following recursive formula:

1. $f(\overline{K}_n) = t^n$, $f(\phi) = 1$ where ϕ is the null graph,
2. $f(G) = xf(G/e) + yf(G - e)$, $e \in E(G)$,

where G/e, $G - e$ denote the graphs obtained from G by *contracting*, *deleting* an edge $e \in E(G)$ respectively.

a) (10 points) Show that $f(G)$ is well-defined (*i.e.*, not depend on the choice of an edge e in definition), and give straightforward interpretations for $f(G; 1, 1, 1)$, $f(G; 2, 0, 1)$ and for $f(G; t, -1, 1)$ respectively.

b) (10 points) Find $f(K_3; t, x, y)$ and $f(P_4; t, x, y)$ where P_4 is a *path* with 4 vertices.

c) (10 points) Let $W = [w(x, y)]$ be a symmetric matrix over complex numbers indexed by a finite set X, and let

$$Z(G, W) = \sum_{\sigma} \prod_{(a,b) \in E(G)} w(a^{\sigma}, b^{\sigma})$$

where σ runs through all $X^{V(G)}$. Show that $Z(G, J - I)$ is an example of $f(G; t, -1, 1)$ for some t.

IV. (15 points)

a) (5 points) Show that the completement of the *line graph* $L(K_5)$ of K_5 is isomorphic to the graph Γ as shown below.

b) (5 points) Describe as many properties of Γ as possible in terms of its adjacency matrix A; for example, the 3-regularity of Γ can be expressed as $J \cdot A = A \cdot J = 3J$.

c) (5 points) Find the *minimal polynomial* of A (and hence of Γ).