國立交通大學試題紙

科目：分析　課號：　班別：博士班招生　日期：87年6月3日第1頁共3頁

請"V"明☑不可看書，☐可看書

1. Answer the following questions and give brief explanations

① If f is improper Riemann-integrable on $[1,\infty)$, (briefly, $f \in IR[1,\infty)$), is $|f| \in IR[1,\infty)$?

② If $|f| \in IR[1,\infty)$, is $f \in \mathcal{L}(1,\infty)$?

③ Let (f_n) be a sequence of real-valued functions defined on $[1,\infty)$. If $f_n(x) \to f(x)$ a.e., $|f_n(x)| \le g(x)$ a.e. and $g \in IR[1,\infty)$, is $f \in \mathcal{L}(1,\infty)$?

④ Construct a sequence (f_n) of Riemann-integrable functions on $[0,1]$, such that $|f_n(x)| \le 1$ for all $0 \le x \le 1$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} f_n(x) = f(x)$ exists everywhere, but $f(x)$ is not Riemann-integrable on $[0,1]$

2. Let Ω be a bounded open set in R^n and let $1 \le q < r \le p < \infty$.

① Show that the inclusion map $I: L^r(\Omega) \to L^q(\Omega)$ is continuous

② Show that the norms of $L^r(\Omega)$ and of $L^q(\Omega)$ are not equivalent

3. Show that if $f \in \mathcal{L}$ and (E_n) is a sequence of measurable sets with $\lim_{n\to\infty} E_n = E$, $\mu(E) = 0$, then $\lim_{n\to\infty} \int_{E_n} f \, d\mu = 0$

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4 Let Ω be a measure space. $g: X\times R \to R$,
$g(x,r)$ is measurable with respect to x for each $r\in R$
and continuous with respect to r for each $x\in\Omega$.

① Show that $g(x,u(x))$ is a measurable function of x if $u(x)$ is measurable.

② Assume that $|g(x,r)|\le h(x)$, $h\in L^2(\Omega)$.
Show that the map $G: L^2(\Omega)\to L^2(\Omega)$ defined by
$(G(u))(x) = g(x,u(x))$ is continuous

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① State Fubini's theorem for $f\in \mathcal{L}(X\times Y, A\times B, \mu\times\nu)$

② Let X be a compact subset in R^n and let μ denote the Lebesgue measure. Let $K(x,y)\in L^2(X\times X, \mu\times\mu)$.
For any given $g\in L^2(X,\mu)$, consider the operator
$(Tf)(x) = \int K(x,y) f(y)\, d\mu(y)$ and the equation $(*)$ $f(x) = g(x) + \lambda (Tf)(x)$
Show that T is compact (hint: Approximate K by continuous functions) and that if $g=0$ implies $f=0$, then there exists a unique solution of equation $(*)$ for any $g\in L^2(X,\mu)$.

6 Let (x_n) be weakly convergent to $x\in X$, X a normed linear space. Show that

① (x_n) is bounded

② $\|x\| \le \underline{\lim}_n \|x_n\|$

③ Use ② to show that if X is weakly sequentially complete then it is complete.

◎請用深黑色鋼筆或原子筆出題

命題教師簽名：

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7. Let T be defined by $(Tx)(t) = t x(t)$, $(0<t<1)$.

① Is T a bounded linear operator in $L^2(0,1)$? Find the norm of the operator T.

② Is T a compact operator in $L^2(0,1)$? Prove or disprove by counter-examples.

8. ① Show that the norm of a Hilbert space is strictly convex. (i.e. $\|x\| = \|y\| = 1$, $\|x+y\| = 2 \Rightarrow x = y$)

② Let $f \in L^2(0,1)$. Show that for any positive integer n there exists a unique polynomial p_n of degree $\le n$ such that $\|f-p\| \ge \|f-p_n\|$ for all polynomial p of degree $\le n$.

9. Let $K(x,y)$ be continuous on $G \times G$, G a compact subset in R^n. Prove that the operator T defined by $(Tf)(x) = \int_G K(x,y) f(y)\, dy$ is a compact linear operator in $C(G)$.