NATIONAL CHIAO TUNG UNIVERSITY
DEPARTMENT OF APPLIED MATHEMATICS
PH.D. ANALYSIS ENTRANCE EXAM

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Instruction: The problems are not arranged in accordance with their difficulty levels. Please read all problems first and do the ones that are easiest for you. There are a few hints given for some problems, read them only if you have no idea how to begin with.

Problem 1. *(10 points) Let f be a real valued function defined on $[0,1]$ which is Riemann integrable over $[b,1], \forall b \in (0,1)$. Suppose farther that f is a bounded function. Prove or disprove that f is Riemann integrable over $[0,1]$.*

Problem 2. *(7 points each) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function whose partial derivatives of order ≤ 2 are defined and continuous everywhere.*

(i) What is the Taylor polynomial of degree 1 for f at $a \in \mathbb{R}^2$ and its remainder?

(ii) Let $a \in \mathbb{R}^n$ be a critical point of f (i.e. $\frac{\partial f}{\partial x_i}(a) = 0$, for $i = 1,2$). Prove that f attains its local minimum at $x = a$ if the Hessian matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

is positive definite at $x = a$ (a square matrix A is positive definite if $v^T Av > 0, \forall$ column vector $\mathbb{R}^2$ where v^T is the transpose matrix of v).

(iii) Suppose the Hessian matrix of f is positive definite at all $a \in \mathbb{R}^2$. Prove that f has at most one critical point.

Problem 3. *(5 points each) Suppose we have a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$.*

(i) State the definition that f is differentiable at a point $a \in \mathbb{R}^n$.

(ii) Suppose all the partial derivatives of f at $a \in \mathbb{R}^n$ exist. Is f differentiable at a? Justify your answer!

Problem 4. *(10 points) Let f be a Lebesgue measurable function defined on $\mathbb{R}^n$ and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a transformation defined by $Ax = Mx + b$ for some invertible $n \times n$ matrix M and a fixed $b \in \mathbb{R}^n$. Show that $f \circ A$ is also Lebesgue measurable.*

Problem 5. *(10 points) Compute*

$$\int_{\mathbb{R}^2} e^{-x^2-y^2}\, dx \times dy,$$

where $dx \times dy$ is the product measure of two copies of Lebesgue measure on $\mathbb{R}$. Justify your steps.

Problem 6. *(9 points) Let $f \in C([0,1])$ such that*

$$\int_{[0,1]} f\varphi' = 0, \qquad \forall \varphi \in C^1([0,1]) \text{ with } \varphi(0) = \varphi(1) = 0.$$

Show that f is a constant function and find the constant without specify the value of f at specific point in $[0,1]$, if that is possible. (**Hint:** *Integration by parts?!)*

Problem 7. *(10 points each) The Lebesgue's bounded convergence theorem states: Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions defined on a set E and*

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \forall x \in E.$$

If (a) $|f_n(x)| \leq M$ for all x and n, and
(b) E is a set of finite measure, then

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

(i) What happen if condition (a) is omitted?
(ii) What happen if condition (b) is omitted?
Verify your assertions.

Problem 8. *(10 points) Is $L^p(\mathbb{R}^n)$, for $1 < p < \infty$ and $p \neq 2$, a Hilbert space with respect to its L^p norm structure? Verify your assertion!* (**Hint:** *Parallelogram law?!)*