八十九學年度博士班入學考試
分析

PH.D. ANALYSIS ENTRANCE EXAM
JUNE 2, 2000

89.6.2

1(20 points) (a) let $f : R^2 \to R$ be C^2. Show that f satisfies $\partial^2 f/\partial x\partial y(x,y) \equiv 0$ if and only if there exist C^2 functions $g, h : R \to R$ such that $f(x,y) = g(x) + h(y)$.

(b) Let $f : R^2 \to R$ be C^2. Show that f satisfies $\partial^2/\partial t^2 f(x,t) = c^2(\partial^2/\partial x^2)f(x,t)$ (vibrating string equation) if and only if there exist C^2 functions $g, h : R \to R$ such that $f(x,t) = g(x+ct) + h(x-ct)$. Here c is a constant (the speed of sound). (Hint: make a change of variable to reduce to problem (a).)

2(20 points) (a) Let g be a nonnegative measurable fonction on [0,1]. Show that $\log\int g(t)dt \geq \int \log(g(t))dt$ whenever the right side is defined.

(b) Use Fubini's theorem to prove that $\int_{R^n} e^{-|x|^2}dx = \pi^{\frac{n}{2}}$. (For n=1, write $(\int_{-\infty}^{\infty} e^{-x^2}dx)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dxdy$ and use polar coordinates.)

3(20 points) Let $\{f_n\}$ be a sequence of functions in $L^p, 1 < p < \infty$, which converge almost everywhere to a function f in L^p, and suppose that there is a constant M such that $\|f_n\|_p \leq M$ for all n. Show that $\int f_n g \to \int fg$ for all $g \in L^q, 1/p + 1/q = 1$. Is the result true for $p = 1$?

4(20 points) (a) For $1 \leq p < \infty$, we denote by l^p the space of all sequences $\{a_n\}_{n=1}^{\infty}$ such that $\|\{a_n\}\|_p \equiv (\sum_{n=1}^{\infty} |a_n|^p)^{1/p} < \infty$. Show that l^p is separable. (A metric space X is said to be separable if it has a countable dense subset.)

(b) Let l^∞ be the space of all bounded sequences with the sup norm. Show that l^∞ is not separable.

5(20 points) Let S be a linear subspace of $C[0,1]$ which is closed as a subspace of $L^2[0,1]$.

(a) Show that S is a closed subspace of $C[0,1]$.

(b) Show that there is a constant M such that for all $f \in S$ we have $\|f\|_2 \leq \|f\|_\infty$ and $\|f\|_\infty \leq M\|f\|_2$.

(c) Show that for each $y \in [0,1]$ there is a function k_y in L^2 such that for each $f \in S$ we have $f(y) = \int k_y(x)f(x)dx$.