八十九學年度博士班入學考試離散數學

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Discrete Mathematics (Entering Examination for the Ph.D Program)
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1. (20%) For positive integers $n \geq k \geq 1$, define $G_{n,k} = (V_{n,k}, E_{n,k})$ to be the graph with vertex set $V_{n,k} = \{a = a_1a_2 \ldots a_n : a_i \text{ is } 0 \text{ or } 1 \text{ for } 1 \leq i \leq n\}$ and edge set $E_{n,k} = \{ab : a_i \neq b_i \text{ for exactly } k \text{ indices } i\}$.

(a) Determine $|V_{n,k}|$ and $|E_{n,k}|$.

(b) What are the degrees of the vertices in the graph $G_{n,k}$?

(c) Determine the number of (connected) components of the graph $G_{n,k}$. (Hints. Consider three cases: $n = k$, $n > k$ with k even, $n > k$ with k odd.)

(d) For which n and k, the graph $G_{n,k}$ is bipartite?

2. (20%) Suppose $(d_1, d_2, \ldots, d_n)$ is a sequence of nonnegative integers.

(a) Prove that for any graph $G = (V, E)$, we have $\sum_{v \in V} \deg(v) = 2|E|$.

(b) Prove that $(d_1, d_2, \ldots, d_n)$ is the degree sequence of a graph (with loop and multiple edges allowed) if and only if $\sum_{i=1}^{n} d_i$ is even.

(c) Suppose $n \geq 2$. Prove that $(d_1, d_2, \ldots, d_n)$ is the degree sequence of a tree if and only if $\sum_{i=1}^{n} d_i = 2n - 2$ and all d_i are positive.

3. (20%) A *system of distinct representatives* (SDR) of a sequence $(A_1, A_2, \ldots, A_n)$ of sets is a sequence $(a_1, a_2, \ldots, a_n)$ of n distinct elements such that $a_i \in A_i$ for $1 \leq i \leq n$. The well-known marriage theorem says that $(A_1, A_2, \ldots, A_n)$ has an SDR if and only if the union of any k sets A_i contains at least k elements. Use this theorem, or any method you fell comfortable, to prove the following assertions.

(a) A *doubly stochastic matrix* is a nonnegative real matrix who row sums and column sums are 1. A *permutation matrix* is a 0-1 doubly stochastic matrix. Prove that For any doubly stochastic matrix D, there exist permutation matrices $P_1, P_2, \ldots, P_k$ and positive numbers $c_1, c_2, \ldots, c_k$, summing up to 1, such that $D = c_1P_1 + c_2P_2 + \ldots + c_kP_k$.

(b) An $r \times n$ *Latin rectangle* ($r \leq n$) is an $r \times n$ matrix that has numbers $1, 2, \ldots, n$ as entries such that no number appears more than once in the same row or the same column. Prove that it is always possible to append $n - r$ rows to an $r \times n$ Latin rectangle to form an $n \times n$ Latin square.

4. (20%) Given a sequence $x_1, x_2, \ldots, x_n$ of real numbers (not necessarily positive), the maximum consecutive subsequence problem is to find a subsequence $x_i, x_{i+1}, \ldots, x_j$ of consecutive elements such that the sum of the numbers in it is maximum over all subsequences of consecutive elements. The sum of the empty subsequence is defined as 0. For instance, in the sequence $(2, -3, 1.5, -1, 3, -2, -3, 3)$ the maximum consecutive subsequence is $(1.5, -1, 3)$.

(a) Implement the following method, which solves the problem, more formally and determine its time complexity.

$$s_{ij} = x_i + x_{i+1} + \ldots + x_j \text{ for } 1 \le i \le j \le n;$$
$$\text{max-sum} = \max\{s_{ij} : 1 \le i \le j \le n\}.$$

(b) Revise the above algorithm so that it outputs not only the value max-sum but also the subsequence itself.

(c) Write an algorithm for the problem with a better time complexity than that in part (a). Most favorable one is of $O(n)$ time. Your algorithm must also be able to do the same job as in (b).

5. (20%) A real-valued matrix is called *totally unimodular* if any of its submatrices has determinate 1 or 0 or -1.

(a) Prove that if M is totally unimodular then so is its transpose M^{T}.

(b) Prove that if M is totally unimodular then so is the matrix obtained from M by multiplying -1 to a row.

(c) Suppose A is the vertex-edge incidence matrix of a digraph $G = (V, E)$, i.e., A is a $|V| \times |E|$ matrix define by

$$a_{ij} = \begin{cases} 1, & \text{if edge } j = ii'; \\ -1, & \text{if edge } j = i'i; \\ 0, & \text{otherwise.} \end{cases}$$

Prove that A is totally unimodular.

(d) Suppose A is the vertex-edge incidence matrix of a graph $G = (V, E)$, i.e., A is a $|V| \times |E|$ matrix define by

$$a_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ is incident to edge } j; \\ 0, & \text{otherwise.} \end{cases}$$

Prove that A is totally unimodular if and only if G is bipartite.