Analysis

Ph.D. Entrance Exam (June 1, 2001)

(10%) 1. Let f be a bounded linear functional on a subspace of ℓ^2. Prove that f has a norm-preserving extension to ℓ^2. Is such an extension unique? Either prove it or give a counter-example.

(20%) 2. (a) Let $\{f_n\}$ be a sequence of measurable functions on [0, 1]. Prove that if $\sum_n \int_0^1 |f_n| < \infty$, then $\sum_n f_n$ converges absolutely a.e. on [0, 1].

(b) Let $\{r_n\}$ be a sequence of all rational numbers in [0, 1] and let $\{a_n\}$ be such that $\sum_n |a_n| < \infty$. Use (a) to prove that

$$\sum_n \frac{a_n}{\sqrt{|x - r_n|}}$$

converges absolutely a.e. on [0, 1].

(20%) 3. (a) Let M be a subset of C[0, 1]. Give a necessary and sufficient condition on elements in M in order that it be compact. Explain any technical terms in your statement.

(b) Use (a) to prove that $M = \{f : [0, 1] \longrightarrow [0, 1] : |f(x) - f(y)| \leq |x - y|/2$ for all x, y in $[0, 1]\}$ is compact in C[0, 1].

(20%) 4. Let $1 \leq p < q < \infty$.

(a) Determine which of ℓ^p and ℓ^q contains the other. Prove your assertion.

(b) Do the same for $L^p(0, 1)$ and $L^q(0, 1)$.

(30%) 5. (a) Give a complete statement of the Stone-Weierstrass theorem for algebras of real-valued functions on X. Explain any technical terms in your statement.

(b) Do the same for complex-valued functions. Give an example to show that the real version does not hold in the complex case.

(c) Use the Stone-Weierstrass theorem to evaluate the limit

$$\lim_{n \to \infty} (n + 1)\int_0^1 x^n f(x)\,dx$$

for any real-valued continuous function f on [0, 1].