博士班入学考試
分析

Ph.D Analysis Entrance Exam

May 31, 2002

1.(10 points)

For any two real sequences $\{a_n\}$, $\{b_n\}$,

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided that the sum on the right is not of the form $\infty - \infty$.

2.(8 points)

Let X and Y be metric spaces and consider the mapping $f : E \to Y$, where $E \subseteq X$. Prove that $f : E \to Y$ is continuous if and only if for each open set V in Y there exists an open set O in X such that $f^{-1}(V) = O \cap E$.

3. (12 points)

Let K be a compact metric space and $\{f_n\}$ a sequence of continuous functions to a metric space Y which converges at each point of K to a function f. Prove that $\{f_n\}$ converges uniformly on K if and only if $\{f_n\}$ is equicontinuous on K.

4. (10 points)

Suppose that f is a Riemann integrable function on $[a,b]$, prove that there is a polynomial P_n such that $\lim_{n\to\infty} \int_a^b |f(x) - P_n(x)|^2 dx = 0$.

5. (10 points)

Let X and Y be compact spaces, prove that for each continuous real-valued function f on $X \times Y$ and each $\varepsilon > 0$, there are continuous real-valued functions $g_1, g_2, \ldots, g_n$ on X and $h_1, h_2, \ldots, h_n$ on Y such that for each $(x,y) \in X \times Y$ we have $|f(x,y) - \sum_{i=1}^n g_i(x)h_i(y)| < \varepsilon$.

6.(10 points)

Let X be a complete metric space, and suppose that the mapping $T : X \to X$ is a contraction. (that is, there is a number c with $0 \leq c < 1$ such that $d(T(x), T(y)) \leq cd(x,y)$.) Prove that there is a unique point x in X such that $T(x) = x$.

7. (15 points) Prove that $L^p[0,1]$ is a Banach space.

8. (25 points)

(a) Let $E \subseteq \Re$ be a measurable set of finite measure and let $\{f_n\}$ and f be measurable real-valued functions defined on E. Prove that $\{f_n\}$ converges to f in measure if and only if given $\varepsilon > 0$ there is a set E_ε, with $m(E_\varepsilon) < \varepsilon$ such that f_n converges to f uniformly on $E \setminus E_\varepsilon$.

(b) Comment on the result when E in (a) is of infinite measure and justify your answer.