博士班入學考試

Ph.D. Entrance Exam: Analysis

May 27, 2003 分析

(20%) 1. Let $f_n : [0, 1] \longrightarrow [0, \infty)$, $n = 1, 2, \ldots$, be a sequence of Lebesgue measurable functions.

(a) Give an example to show that $f_n \longrightarrow f$ a.e. as $n \longrightarrow \infty$ does not imply that $\int_0^1 f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx$. (5%)

(b) What general relation (equality, inequality?) on the integrals of f_n and f can be concluded from $f_n \longrightarrow f$ a.e.?

(c) Give at least two sets of different conditions on the f_n's for which we can conclude $\int_0^1 f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx$ from $f_n \longrightarrow f$ a.e. (10%)

(10%) 2. Prove that if $f : [a, b] \longrightarrow \mathbb{R}$ is Lebesgue integrable and $\int_a^x f(t)\,dt = 0$ for all x in $[a, b]$, then $f = 0$ a.e. on $[a, b]$.
(You are not supposed to prove this by using a more general theorem.)

(20%) 3. Let f be a function in $L^p[0, 1]$ with $1 \leq p < \infty$ and let $F(x) = \int_0^x f(t)\,dt$ for x in $[0, 1]$.

(a) Prove that $\|F\|_p \leq (1/p)^{1/p} \|f\|_p$. (10%)

(b) Give a necessary and sufficient condition on f for which in (a) the equality holds and prove your assertion. (10%)

(20%) 4. (a) Prove that a linear functional on a normed space is bounded if and only if its kernel is closed. (10%)

(b) Determine to what extent (a) can be generalized to linear transformations between two normed spaces. Justify your assertions with proofs and examples. (10%)

(10%) 5. Let M be a linear manifold of a normed space. Prove that M is weakly closed if and only if it is closed in norm.

(20%) 6. (a) Prove that if $f : [0, 2\pi] \longrightarrow \mathbb{R}$ is Lebesgue integrable, then $\lim_{n\to\infty} \int_0^{2\pi} f(x) \cos nx\,dx = 0$. (10%)

(b) Let $n_1 < n_2 < \ldots$ be positive integers. Use (a) to prove that the set $E = \{x \in [0, 2\pi] : \lim_{k\to\infty} \cos(n_k x) \text{ exists}\}$ has Lebesgue measure zero. (10%)

(Hint: Consider $\lim_{k\to\infty} \int_E \sin^2 n_k x\,dx$.)