NATIONAL CHIAO TUNG UNIVERSITY
DEPARTMENT OF APPLIED MATHEMATICS
PH.D. ANALYSIS ENTRANCE EXAM

May 19, 2004

Instruction: The problems are not arranged in accordance with their difficulty levels. Please read all the problems first and do the ones that are easiest for you. There are a few hints given for some problems, read them only if you have no idea how to begin with.

1. (a) (10 points) Give a complete statement of Ascoli-Arzelá theorem for real-valued functions.

   (b) (15 points) A real-valued function f on $[0,1]$ is said to be Hölder continuous of order α if there is a constant C such that

   $$|f(x)-f(y)| \le C|x-y|^\alpha, \quad \text{for all } x, y \in [0,1].$$

   Define

   $$\|f\|_\alpha = \max_{x\in[0,1]} |f(x)| + \sup_{x,y\in[0,1], x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

   Show that for $\alpha \in (0,1]$, the set S of functions with $\|f\|_\alpha \le 1$ is a compact subset of $C[0,1]$, the set of all continuous functions defined on $[0,1]$.

2. (15 points) Let $a = \{a_k\}, b = \{b_k\}$ be two sequences of real numbers such that

   $$\|a\|_p = \left(\sum_{k=0}^{\infty} |a_k|^p\right)^{\frac{1}{p}}, \qquad \|b\|_q = \left(\sum_{k=0}^{\infty} |b_k|^q\right)^{\frac{1}{q}} < \infty,$$

   where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Define a sequence $c = a * b = \{c_k\}$ by $c_k = \sum_{j=0}^{k} a_j b_{k-j}$ for $k = 0, 1, 2, \cdots$. Prove that c is an absolute convergent sequence.
   (**Hint:** Hölder's inequality?!)

3. (15 points) Prove the *Riemann-Lebesgue Theorem:* If f is an integrable function on $\mathbb{R}$, then $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \sin nx \, dx = 0$.
   (**Hint:** Start with when f is a simple function.)

4. For each of the following statements give an example of a sequence of real-valued functions $\{f_k\}$ and/or a real-valued function f defined on a Lebesgue measurable subset Ω in $\mathbb{R}^n$ such that each of the statements holds:

   (a) (10 points) $f_k, f \in L^p(\Omega)$ for all k, with $1 < p < \infty$ fixed, and $\|f_k - f\|_p \longrightarrow 0$, but f_k does not converges to f almost everywhere in Ω.

   (b) (10 points) $f \in L^3(\Omega) \setminus L^2(\Omega)$.

   (c) (10 points) $f \in L^2(\Omega) \setminus L^3(\Omega)$.

5. (15 points) Compute $\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{\frac{x}{3}} \, dx$ and justify your answer.