交通大學應用數學系
93 學年度博士班入學考試

離散數學

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1. (20%) Let $B \subseteq \{(i, j) \mid 1 \le i, j \le n\}$, and define $G_\sigma = \{(i, i^\sigma) \mid 1 \le i \le n\}$ for $\sigma \in S_n$, the family of all permutations of $\{1, 2, ..., n\}$. Let

 N_j = the number of $\sigma \in S_n$ with $|B \cap G_\sigma| = j$ for $1 \le j \le n$, and

 r_k = the number of k-subsets of B such that no two of them have a common coordinate.

 a. Find N_0 as straight as possible whenever $B = \{(i, i) \mid 1 \le i \le n\}$, denote it by D_n.

 b. Justify the expression $\sum_{j=k}^{n} \binom{j}{k} N_j = r_k (n-k)!$, and then show that $\sum_{j=0}^{n} N_j x^j = \sum_{k=0}^{n} r_k (n-k)!(x-1)^k$, it follows that $N_0 = \sum_{k=0}^{n} (-1)^k r_k (n-k)!$.

 c. Derive r_k whenever $B = \{(i, i), (i, i+1 (\mathrm{mod}\, n)) \mid 1 \le i \le n\}$.

2. (20%) Let $(x)_k = x(x-1)\cdots(x-k+1)$, then both $\{1(=x^0), x(=x^1), x^2, ..., x^n\}$ and $\{1(=(x)_0), x(=(x)_1), (x)_2, ..., (x)_n\}$ are basis of the vector space of all polynomials with degrees at most n over R, and hence there are coefficients $S(n, k)$ and $s(n, k)$ such that $x^n = \sum_{k=0}^{n} S(n, k)(x)_k$, and $(x)_n = \sum_{k=0}^{n} s(n, k) x^k$.

 a. Find combinatorial interpretations for $S(n, k)$, $c(n,k) = (-1)^{n-k} s(n,k)$ respectively.

 b. Find a closed form for $\sum_{n \ge k} S(n, k) \frac{x^n}{n!}$.

 c. For sequences $a_0, a_1, ..., a_n, ...$ and $b_0, b_1, ..., b_n, ...$ of complex numbers, show that the following are equivalent:

 (*i*) $b_n = \sum_{k=0}^{n} S(n,k) a_k$ for all $n \ge 0$, (*ii*) $a_n = \sum_{k=0}^{n} s(n,k) b_k$ for all $n \ge 0$.

3. (20%) A *system of distinct representatives* (*SDR*) of the system ($A_1, A_2, \ldots, A_n$) of sets is a sequence $(a_1, a_2, ..., a_n)$ of pairwise distinct elements with $a_i \in A_i$ for each $1 \le i \le n$. The Hall's Theorem shows that the system ($A_1, A_2, \ldots, A_n$) has an *SDR* if and only if the *Hall's condition* $|\mathcal{U}(I)| \ge |I|$ holds for each $I \subseteq \{1, 2, ..., n\}$, where $\mathcal{U}(I) = \bigcup_{i \in I} A_i$.

3. (continuous)

a. Find the number of all *SDR* over the system $(A_1, A_2, \ldots, A_n)$ where $A_1 = \{2,\ldots,n\}$ and $A_i = \{1, 2,\ldots, i-1, i+1,\ldots, n\}$ for each $2 \le i \le n$.

b. Show that a $k \times n$ latin rectangle, with $k < n$, can be extended to a latin square of order n by adding $n-k$ new rows.

c. For a fixed d, what can we conclude if $|\mathcal{U}(I)| \ge |I| - d$ for each $I \subseteq \{1, 2,\ldots, n\}$? Justify your answer if the Hall's Theorem is assumed.

4. (25%) The *Kneser graph* $G = K(n,d)$ is defined on the family of all d-elements subsets of $\{1,2,\ldots,n\}$ such that A and B are adjacent if and only if A and B are disjoint.

a. Is the complement of the line graph of the complete graph K_5 isomorphic to a Kneser graph $K(n,d)$ for some n and d? Justify your answer.

b. Find the chromatic numbers $\chi(K(5,2))$ and $\chi(K(6,2))$ respectively.

c. Find conditions for A, B such that the distance $d(A,B) = i$ in the graph $K(n,d)$ for $i = 1, 2, 3$.

d. Determine the smallest 3-regular graphs with girths 4 and 5 respectively.

5. (15%) Start with $G_3 = C_5$ the 5-cycle, and for a given graph G_n, the new graph G_{n+1} is defined on $V(G_n) \cup V(G_n)' \cup \{z_{n+1}\}$, where the vertices $v' \in V(G_n)'$ correspond bijectively to $v \in V(G_n)$, and z_{n+1} is a single other vertex. The edges of G_{n+1} consists of 3 classes: all edges of G_n, each vertex $v' \in V(G_n)'$ is joint to precisely the neighbors of v in G_n, and z_{n+1} is adjacent to all $v' \in V(G_n)'$. The graphs $G_3 = C_5$ and G_4 are given as shown.

a. Find the girth of the graph G_4, and show that the chromatic number $\chi(G_4)$ is 4.

b. Show that the chromatic number $\chi(G_n)$ of G_n is n in general.

Note: The *girth* of a graph is defined to be the length of the shortest cycle in it.