2004 NCTU Department of Applied Mathematics

Ph.D. Entrance Examination: Linear Algebra

Answer all 6 questions (total 40 points).

All vector spaces are over $\mathbf{R}$ or $\mathbf{C}$ (real or complex numbers). For $F = \mathbf{R}, \mathbf{C}$, we always write the vectors in F^n as column vectors. So if A is an $n \times n$ matrix and $v \in F^n$, then $Av \in F^n$. Let $e_1, ..., e_n$ be the natural basis of F^n.

QUESTION 1 (6 points) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Find all the eigenvalues and eigenvectors of A. Show method.

(b) Find a nonzero polynomial $p(x)$ such that $p(A) = 0$, with the degree of $p(x)$ as low as possible. No proof is needed.

(c) Let B be another 3×3 matrix, and $\det B = 3$. Find the determinant of $2A^{-2}B^3$. No proof is needed.

QUESTION 2 (7 points) Let A be a 3×3 orthogonal matrix, and $Ae_1 = e_1$.

(a) What are the possible determinants of A? For each possible determinant, give an example for A.

(b) If B is a 3×3 symmetric matrix, what can you say about ABA^{-1}? Explain.

(c) What are the possibilities for $e_2 \times Ae_2$? Explain. (here $\times$ is the usual cross product in $\mathbf{R}^3$)

QUESTION 3 (7 points) Let V be the $n \times n$ complex matrices. It is a complex vector space under the usual matrix addition and scalar multiplication. Let

$$W = \{A \in V\ ;\ A \text{ is Hermitian and trace } A = 0\}.$$

Remark: Recall that the elementary $n \times n$ matrix E_{rs} has 1 at the (r, s)-entry and 0 elsewhere. For this question, it is convenient to use these E_{rs} to help describe your answer, and avoid messy big matrices.

(a) Is W a complex linear subspace of V? Answer *yes* or *no*. If *yes*, no proof is needed, but write down a basis for this complex subspace and find its dimension. If *no*, explain why not.

(b) Repeat part (a) by replacing every word "complex" with "real".

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QUESTION 4 (6 points) Let V be a real vector space with inner product $\langle , \rangle$. In each of (a) and (b), decide if L is a linear transformation from the vector space $V \oplus V$ to the vector space $\mathbf{R}$, and explain your answer. For all $v, w \in V$, define:

(a) $L(v, w) = \langle v, w \rangle$.

(b) $L(v, w) = \langle v, z \rangle$, where $z \in V$ is a fixed vector.

QUESTION 5 (8 points) Let $T : \mathbf{R}^9 \longrightarrow \mathbf{R}^9$ be a linear transformation, and $T^2 = 0$. Let $I = \{v_1, ..., v_k\}$ be a basis for the image of T.

(a) Show that there exists some vector in the kernel of T which is not in the image of T.

(b) What is the biggest possible value of k?

(c) Extend I to a basis $v_1, ..., v_k, \underline{v_{k+1}, ..., v_9}$ of $\mathbf{R}^9$. Describe the matrix representation of T with respect to this basis.

(d) Let A be the matrix representation of T with respect to an arbitrary basis of $\mathbf{R}^9$. Find the trace of A. Explain. (Hint: Use result of part (c) to help)

QUESTION 6 (6 points) Let X and Y be linear subspaces of a vector space V. Are the following conditions (a) and (b) equivalent? Explain your answer.

(a) either $X \subset Y$ or $Y \subset X$.

(b) $X \cup Y$ is a linear subspace of V.

T $R(T)$ $a_1 T(V_1) + \cdots T(V_k)$

(the end)