Ph.D. Entrance Exam: Analysis

May 24, 2005

(20%) 1. Let $f : X \longrightarrow \mathbb{R}$ be a measurable function on the measure space $(X, \mathcal{A}, u)$.

(a) If f is integrable, is f^2 integrable? (10%)

(b) If f^2 is integrable, is f integrable? (10%)

In each case, either give a proof or give a counterexample.

(20%) 2. (a) If $f : [a, b] \longrightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$, then prove that f is of bounded variation over there. (10%)

(b) Is the converse of (a) true? Either prove it or give a counterexample. (10%)

(20%) 3. (a) Give the complete statement of Ascoli-Arzelá theorem. Explain all technical terms in your statement. (10%)

(b) Use this theorem to prove the compactness of the set

$$\left\{ f \in C[0, 1] : \max_x |f(x)| + \sup_{x,y} \frac{|f(x) - f(y)|}{\sqrt{|x - y|}} \leq 1 \right\}$$

in $C[0, 1]$. (10%)

(20%) 4. Let $\{x_n\}$ be a sequence of vectors in a normed space X which converges weakly to x in X.

(a) Prove that $\sup_n \|x_n\| < \infty$. (5%)

(b) Prove that $\|x\| \leq \underline{\lim}_n \|x_n\|$. (5%)

(c) Prove that x belongs to the closed linear subspace generated by the x_n's. (5%)

(d) Does $\|x_n\|$ necessarily converge to $\|x\|$? Either give a proof or give a counterexample. (5%)

(20%) 5. (a) Find all extreme points of the closed unit ball $S = \{f \in L^1[0, 1] : \|f\|_1 \leq 1\}$ of $L^1[0, 1]$. (10%)

(b) Use (a) to prove that $L^1[0, 1]$ is not the dual of any normed linear space. (10%)