Discrete Mathematics

(The Entrance Examination for the Ph.D. Program)

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1. (20%)

(a) A sequence consists of 80 balls, each colored RED or BLUE. The number of BLUE balls is 30. Prove that there are at least two BLUE balls in the sequence either three or six balls apart (for example, balls at positions 21 and 24 are three balls apart; balls at positions 52 and 58 are six balls apart).

(b) Let f be a one-to-one and onto function from $S = \{1, 2, \cdots, n\}$ to S. Let $f^i = f \circ f \circ \cdots \circ f$ be the i-fold composition of f with itself. Prove that there exists an integer i such that $f^i(x) = x \ \forall x \in S$.

2. (20%)

(a) Find the number of permutations $p_1p_2p_3p_4p_5p_6$ of $\{1, 2, 3, 4, 5, 6\}$ such that

$$p_1 \neq 1, 4;\ p_2 \neq 2, 3, 4;\ p_3 \neq 5 \text{ and } p_4 \neq 4, 6.$$

(b) Find the number of ways of choosing 12 elements from

$$\{a, a, a, b, b, b, b, c, c, c, c, d, d, d, d, d\}.$$

3. (20%)

(a) A sequence $a_1, a_2, \cdots, a_{2n}$ formed by n boys and n girls is *unacceptable* if it contains a subsequence $a_1, a_2, \cdots, a_k$ such that the number of boys in the subsequence is greater than the number of girls in the subsequence. Let b_n denote the number of unacceptable sequences formed by n boys and n girls. Find b_2 and determine an explicit formula for b_n.

(b) Let c_n be the number of sequences $s_1, s_2, \cdots, s_n, s_{n+1}, s_{n+2}, \cdots, s_{2n}$ formed by the $2n$ integers $1, 2, \cdots, 2n$ (each of the $2n$ integers must appear exactly once) such that

$$s_1 < s_2 < \cdots < s_n,$$

$$s_{n+1} < s_{n+2} < \cdots < s_{2n},$$

and

$$s_1 < s_{n+1},\ s_2 < s_{n+2},\ \cdots, s_n < s_{2n}.$$

Find c_2 and determine an explicit formula for c_n.

4. (20%)

(a) Let G be a simple, directed graph and suppose the undirected graph obtained from G by ignoring the direction of the edges is connected. A vertex v of G has *odd (even) parity* if the number of edges leaving v is odd (even). Prove that if u and v are vertices in G having odd parity, then it is possible to change the direction of certain edges in G so that u and v have even parity and the other vertices have their parity unchanged.

(b) Prove that the maximum number of edges in a simple, disconnected undirected graph with n vertices is $\binom{n-1}{2}$.

5. (20%) A *tree* is a simple, undirected graph satisfying the property that if u and v are two vertices in the graph, then there is a unique path from u to v.

(a) Use the above definition to prove that T is a tree if and only if T is connected and adding an edge between any two vertices results in exactly one cycle.

(b) Let G be a simple, undirected graph with a weight function assigned on the edges. How many minimum weight spanning trees can G have if each edge has a different weight? How many can G have if exactly two edges have the same weight? Justify your answer.