95 年度應用數學系乙組博士班入學考題
這份試卷有 5 大題, 共計 100 分. 證明題要詳述完整才能得全部分數

1. (a) [5 points] How many integers n for $0 \le n \le 99999$ do not have their digits each of $2, 5,$ and 8.

   (b) [5 points] How many integers n for $0 \le n \le 99999$ have among their digits each of $2, 5,$ and 8. (could have other numbers)

   (c) [5 points] How many integers n for $0 \le n \le 99999$ have at least one of their digits from $2, 5,$ or 8.

   (d) [5 points] How many integers n for $0 \le n \le 99999$ have at least one of 2, 5, or 8 not appearing in their digits.

2. [13 points] From the integers $1, 2, \ldots, 200$, we choose 101 integers. Show that among the integers chosen there are two such that one of them is divisible by the other.

3. Let $n > k \ge 4$ denote two integers. Let m_k denote the least number of 3-subsets of C_1 which are not contained in each of C_2 and C_3, among all the triple C_1, C_2, C_3 of distinct k-subsets of $\{1, 2, \ldots, n\}$. That is

$$m_k := \text{Min} \quad | \{ S \mid S \subseteq C_1, S \not\subseteq C_2, S \not\subseteq C_3, |S| = 3\} |,$$

where the minimum is taking from all the triples C_1, C_2, C_3 of distinct k-subsets of $\{1, 2, \ldots, n\}$.

   (a) [5 points] Determine the number m_4.

   (b) [10 points] Determine the number m_k.

   (c) [10 points] Prove your answer in (b).

4. [10 points] Let G denote a simple graph with each vertex of even degree. Show the edges of G are partitioned into cycles.

5. Let $t, q_1, q_2, \ldots, q_k$ be positive integers. The *Ramsey number* $r_t(q_1, q_2, \ldots, q_k)$ is the smallest integer n that satisfies the following property:

   If each of the t-subsets of $\{1, 2, \ldots, n\}$ is assigned one of k colors $c_1, c_2, \ldots,$ and c_k, then there exist an integer i and a q_i-subset T of $\{1, 2, \ldots, n\}$ for $1 \le i \le k$ such that all of t-subsets of T are assigned the color c_i.

   (a) [5 points] Determine the number $r_1(q_1, q_2, \ldots, q_k)$ in terms of an expression of the variables $k, q_1, q_2, \ldots, q_k$.

   (b) [5 points] Prove your answer in (a).

   (c) [12 points] Fill the blank spaces (A), (B), (C), (D), (E), and (F) in the proof of $r_2(3,3,3) \le 3r_2(3,3) - 1$.

   Proof. Set $S := \{1, 2, \ldots, \underline{(A)}\}$. Assign each 2-subset of S one of the colors c_1, c_2 or c_3. We want to show there exists a color $k \in \{c_1, c_2, c_3\}$ and a subset $T \subseteq S$ of cardinality $\underline{(B)}$ such that each 2-subset of T is assigned the color k. Fix an element $a \in S$. Then $|S - \{a\}| = \underline{(C)}$. By the pigeonhole principle, there exists a subset T' of cardinality at least $\underline{(D)}$ such that for each $b \in T'$ the 2-subset $\{\underline{(E)}\}$ is assigned the color k' for some color $k' \in \{c_1, c_2, c_3\}$. If there exist distinct $b, b' \in T'$ such that $\{b, b'\}$ is assigned the color k', then set $T = \underline{(F)}$ and $k = k'$ to finish the proof. Otherwise, all 2-subsets of T' are assigned at most 2 different colors. Since $|T'| \ge \underline{(D)}$, there exists a subset $T \subseteq T'$ of cardinality $\underline{(B)}$ such that each 2-subsets of T is assigned a color k for some $k \in \{c_1, c_2, c_3\} - \{k'\}$.

   (d) [10 points] Show $r_2(q_1, q_2) \le r_2(q_1 - 1, q_2) + r_2(q_1, q_2 - 1)$.