95學年度博士班入學考試

Linear Algebra

95.5.16

Notations.

- The notation $M_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices over $\mathbb{R}$, and I_n is the identity matrix in $M_n(\mathbb{R})$.
- For a matrix A, we let A^t denote the transpose of A.

Problems.

1. Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$. Suppose that a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is defined by $T(x, y, z) = (2x + y, 2y + z, 2z)$.

(1) Write down the matrix of T relative to the standard basis. **(2 points.)**

(2) Write down the matrix of T relative to the ordered basis $\{e_3, e_2, e_1\}$. **(2 points.)**

(3) Find a matrix P such that

$$P^{-1}\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} P = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$$

for all real numbers a. **(3 points.)**

(4) Prove that for any given $n \times n$ matrix A, there is a matrix Q such that

$$Q^{-1}AQ = A^t.$$

(That is, A and A^t are similar for all square matrices A.) **(8 points.)**

(5) Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Find a matrix Q such that $Q^{-1}AQ = A^t$. **(10 points.)**

2. For an $n \times n$ matrix A, define

$$\exp A = I_n + \sum_{k=1}^{\infty} \frac{A^k}{k!}.$$

Prove or disprove (by giving counterexamples) the following two assertions.

(1) If A is nilpotent, then so is $\exp A - I_n$. **(8 points.)**

(2) If $\exp A - I_n$ is nilpotent, then so is A. **(7 points.)**

3. Let $V = M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices over $\mathbb{R}$. For a given matrix $A \in M_n(\mathbb{R})$, define a linear operator T_A on V by

$$T_A(B) = AB - BA, \qquad \forall B \in V.$$

(1) Consider the case $n = 3$ and
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$
Determine the eigenvalues of T_A and the associated eigenspaces. Determine also the minimal polynomial of T_A. **(15 points.)**

(2) For general n, consider the family
$$\mathcal{F} = \{T_A : A \in M_n(\mathbb{R}) \text{ are diagonal matrices.}\}$$
of linear operators. Prove that $\mathcal{F}$ is simultaneously diagonalizable. **(10 points.)**

4. Let V be an inner product space of finite dimension n over $\mathbb{R}$. Recall that a linear transformation $T : V \mapsto V$ is called an *isometry* if $\langle Tv_1, Tv_2\rangle = \langle v_1, v_2\rangle$ for all $v_1, v_2 \in V$.

(1) Prove that a linear transformation T is an isometry if and only if its matrix with respect to an orthonormal basis is orthogonal. (An orthogonal matrix is a square matrix M such that $M^tM = I_n$.) **(10 points.)**

(2) Consider the case $V = \mathbb{R}^n$ with the standard inner product. Let v be a vector of unit length, and define a linear transformation T_v by
$$T_v(u) = u - 2\langle u, v\rangle v \qquad \text{for } u \in V.$$
Prove that T_v is an isometry of $\mathbb{R}^n$. (We call such linear transformations *reflections*.) **(5 points.)**

(3) Consider $V = \mathbb{R}^2$ with the standard inner product. Prove that the linear transformation $S_\theta(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$ is an isometry of $\mathbb{R}^2$ for all real numbers θ. (We call such linear transformation *rotations*.) **(3 points.)**

(4) Prove that every isometry of $\mathbb{R}^2$ is either a rotation or a reflection. **(7 points.)**

5. Let $V = M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices over $\mathbb{R}$, and $f : V \mapsto \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$. Prove that f is the trace function. (*Hint*: Consider the cases $A = e_{ij}$, $B = e_{kl}$ for various i, j, k, l, where $\{e_{ij}\}$ is the standard basis for V.) **(10 points.)**