九十六學年度國立交通大學應用數學系博士班入學考

考試科目: 分析

1. (16 points) Let m be the Lebesgue measure on a finite interval $[a, b]$. Prove that

$$L^p([a,b],m) \subsetneqq L^q([a,b],m)$$

for $1 \leq q < p$.

2. (16 points) Let μ be a σ-finite measure, and let $f_1, f_2 \in L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$. If

$$\int f_1 g \, d\mu = \int f_2 g \, d\mu$$

for all $g \in L^p(\mu)$, show that $f_1 = f_2$ μ-a.e.

3. (26 points) Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x,y) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 2y, & \text{if } x \notin \mathbb{Q}, \end{cases}$$

(a) (8 points) Does the integral

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy,$$

exist as iterated Riemann-integral or iterated Lebesgue-integral?

(b) (8 points) Does the integral

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx$$

exist as iterated Riemann-integral or iterated Lebesgue-integral?

(c) (10 points) Is f Lebesgue integrable on $[0,1] \times [0,1]$?

4. (16 points) Let g be an integrable function on $[0.1]$, and suppose that there is a constant M such that

$$\left| \int fg \right| \leq M \|f\|_p$$

for all bounded measurable function f. Show that $g \in L^q$ and $\|g\|_q \leq M$, where $\frac{1}{p} + \frac{1}{q} = 1$.

5. (26 points) Let C be the Cantor set and let F the Cantor function. Consider a function $f : [0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{2}(x + F(x))$.

(a) (8 points) Prove that C is a nowhere dense subset of $[0, 1]$ and $m(C) = 0$.

(b) (6 points) Find a dense subset E of $[0, 1]$ such that $m(E) = 0$.

(c) (12 points) Show that $f(C)$ is a nowhere dense subset of $[0, 1]$ with $m(f(C)) = \frac{1}{2}$.