The entrance exam of the Ph.D. program
Dept. of Applied Mathematics, NCTU

Topics: Analysis May 5, 2009

1. (10 points) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$ be a measurable map. Let $\mathcal{C}$ be the σ-algebra generated by f, that is, $\mathcal{C} = \{f^{-1}(B) | B \in \mathcal{B}\}$. Show that for any $\mathcal{C}$-measurable function $h : X \to \mathbb{R}$, that is, $f^{-1}(B) \in \mathcal{C}$ for any Borel set B in $\mathbb{R}$, there is a measurable function $g : Y \to \mathbb{R}$ such that $h = g \circ f$.

2. (30 points) Let $(f_n)_0^\infty$ be a sequence of functions on $(0, 1)$ defined iteratively by

$$f_0(t) = t \quad \text{for } t \in (0,1), \quad f_{n+1}(t) = \begin{cases} \frac{1}{2} f_n(3t) & \text{if } t \in (0, 1/3) \\ 1/2 & \text{if } t \in [1/3, 2/3] \\ \frac{1}{2}[1 + f_n(3t - 2)] & \text{if } t \in (2/3, 1) \end{cases}$$

   (a) (10 points) Prove that f_n converges pointwise to a continuous nondecreasing function.

   Set $f = \lim_{n \to \infty} f_n$. Let μ be the Lebesgue measure on $(0, 1)$ and μ_f be the measure induced by f, that is,

$$\mu_f(A) = \mu(f^{-1}(A)).$$

   (b) (10 points) Concerning the Lebesgue decomposition of μ_f relative to μ, let μ_f^a and μ_f^s be respectively the absolutely continuous part and the singular part of μ_f. Describe μ_f^a and μ_f^s.

   (c) (10 points) Let g be a function defined by

$$g(s) = \inf\{t | f(t) \geq s\} \quad \forall s \in (0, 1).$$

   Prove that, for any bounded function F, the Riemann Stieltjes integral of F with respect to g exists and

$$\int_0^1 F(t) dg(t) = \int_{(0,1)} F(t) \mu_f(dt).$$

3. (20 points) Let $(X, \mathcal{A}, \mu)$ be a measure space and $L^p(\mu)$ be the L^p-space. For $1 \leq p < q \leq \infty$, find the relation between $L^p(\mu)$ and $L^q(\mu)$ ($\subseteq$, $\supseteq$, $=$ or incomparable) when

   (a) (10 points) μ is a finite measure;

   (b) (10 points) $\mathcal{A}$ is the σ-algebra generated by finite subsets of X and μ is a counting measure on X, that is, $\mu(A)$ is equal to the number of elements in A if A is a finite set and equal to ∞ if A is an infinite set.

4. (20 points) Let f be a measurable function on a measure space $(X, \mathcal{A}, \mu)$. Prove that:

   (a) (10 points) For $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$,

$$\|f\|_p = \sup_{\|g\|_q \leq 1} \int fg d\mu.$$

   (b) (10 points) If μ is finite, then

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p.$$

5. (20 points) Let μ be the Lebesgue measure on $(0, 1)$ and let $\mathcal{C}$ be the set of infinitely differentiable functions on $(0, 1)$ with compact support. Define an inner product on $\mathcal{C}$ as follows

$$\langle f, g \rangle = \int_{(0,1)} f'(x) g'(x) \mu(dx) \quad \forall f, g \in \mathcal{C}.$$

   Show that the completion of $\mathcal{C}$ under $\langle \cdot, \cdot \rangle$ is

$$\left\{ f \middle| f \text{ is absolutely continuous, } f' \in L^2(\mu), \lim_{x \to 0+} f(x) = \lim_{x \to 1^-} f(x) = 0 \right\}.$$