交通大學應用數學系 98 學年度博士班入學考試

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Instructions: There are **5** problems, and 20 points for each. You must provide all necessary details to earn the full credits.

1. (20 %) Let X be a set of n elements, and Y a set of m elements.
   a. Find the numbers of all subsets of X, all subsets A of X with ***even*** $|A|$, and all subsets B of X with ***odd*** $|B|$ respectively? (6 %)
   b. Find the following numbers:
   all functions from X into Y; all ***one to one*** functions from X into Y; and all functions from X ***onto*** Y respectively. (7 %)
   c. Find the number of all subsets of X containing no consecutive elements. (7 %)

2. (20 %) Let n be a positive integer.
   a. Do either *i* or *ii*, but not both:
   i. Find the number of positive integers $a \le n$ relative prime to $n = 42$. (6 %)
   ii. Find the number of positive integers $a \le n$ relative prime to n in general. (12 %)
   b. State the principle used in *b*, and give one of its generalization. (8 %)

3. (20 %)

   a. Find the number of *perfect coverings* of a $2 \times n$ chessboard by dominoes. (10 %)

   b. Let $N_k(n)$ be the number of sequences $(A_1, A_2, \ldots, A_k)$ of subsets of $\{1, 2, \ldots, n\}$ with $A_1 \subset A_2 \subset \ldots \subset A_k \subseteq [n]$, $A_i \neq A_j$ whenever $i \neq j$. Show that

   $N_1(n) = 2^n$, $N_k(n) = (k+1)N_k(n-1)$,

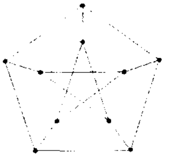
   and explain that

   $$N_k(n) = \sum_{0 \le j \le n} \binom{n}{j} N_k(j) . \ (10\ \%)$$

4. (20 %) Let G be a simple graph.

a. If G has n vertices with at least $\binom{n-1}{2}$ edges, is G connected? (10 %)

b. The *line graph* $L(G)$ of a finite simple graph G is defined on the edge set of the graph G so that two vertices are adjacent in $L(G)$ if they are incident in the graph G as edges. Show that the line graph $L(K_5)$ of the complete graph K_5 of 5 vertices is isomorphic to the complement of the graph G as shown below. (10 %)



5. (20 %) Let M be the *incidence matrix* of a graph G.

a. show that $M' \cdot M = A_L + 2I$ where A_L is the adjacency matrix of the *line graph* $L(G)$ of G; and $M \cdot M' = A + kI$ if G is k-regular. (10 %)

b. find $\det(UV)$ where $U = \begin{bmatrix} xI_n & -M \\ 0 & I_m \end{bmatrix}$ and $V = \begin{bmatrix} I_n & M \\ M' & xI_m \end{bmatrix}$, derive possible information? (10 %)