Ph.D. entrance exam, 2009 – Linear Algebra

1. Let V be an n-dimensional vector space over $\mathbb{R}$ and $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V. (*Symmetric* means $B(u,v) = B(v,u)$ for all $u, v \in V$. *Bilinear* means that B is linear in each of the two variables.)

(a) Let W be a vector subspace of V and let

$$W^\perp = \{u \in V : B(u,v) = 0 \text{ for all } v \in W\}.$$

Prove that if $\dim W = m$, then $\dim W^\perp \geq n - m$. (**5 points.** *Hint:* Choose a basis $\{v_1, \ldots, v_m\}$ for W and consider the map

$$u \longmapsto (B(u, v_1), \ldots, B(u, v_m))$$

from V into $\mathbb{R}^m$.)

(b) Prove that $V = W \oplus W^\perp$ if and only if the restriction of B to W is non-degenerate. (*Non-degenerate* means that $v = 0$ is the only vector of W such that $B(u,v) = 0$ for all $u \in W$.) (**10 points.**)

(c) Recall that an *isometry* with respect to B of V is a function $\varphi : V \to V$ such that $B(\varphi(u), \varphi(v)) = B(u,v)$ for all $u, v \in V$. Prove that if B is non-degenerate, then an isometry φ with respect to B of V is necessarily a linear transformation. (**10 points.**)

(d) Prove that if B is non-degenerate on V, then there is a non-negative integer p with $p \leq n$ and a basis $\{v_1, \ldots, v_n\}$ such that

$$B(v_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq p, \\ -1, & \text{if } p+1 \leq i = j \leq n, \\ 0, & \text{if } i \neq j. \end{cases}$$

(**15 points.** *Hint:* You may want to prove first the assertion that if B' is a non-degenerate symmetric bilinear form on a vector space V' over $\mathbb{R}$, then there exists a vector v such that $B(v,v) \neq 0$.)

(e) Assume that B is non-degenerate and that $\{v_1, \ldots, v_n\}$ is a basis with the property given in Part (d). What can you say about the matrix of an isometry φ with respect to the basis $\{v_1, \ldots, v_n\}$? In particular, show that the determinant of φ must be 1 or -1. (**10 points.** *Hint:* You probably already know that in the case $V = \mathbb{R}^n$ and B is the standard inner product on V, the matrix T of an isometry with respect to an orthonormal basis satisfies $T^t T = I$. How did you prove this?)

2. Let V be the space of all polynomials in x over $\mathbb{R}$ of degree ≤ 2. Let an inner product on V be defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) Find a polynomial $k(x,t)$ in x and t such that

$$f(x) = \int_{-1}^{1} k(x,t) f(t)\,dt$$

for all $f \in V$. (**10 points.**)

(b) Let $T : V \to V$ be the linear transformation defined by $T(a_2x^2 + a_1x + a_0) = 2a_2x + a_1$. Find the linear transformation T^* such that $\langle T(f), g\rangle = \langle f, T^*(g)\rangle$ for all $f, g \in V$. (**10 points.**)

3. Let

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(The characteristic polynomials are both $(x-1)^3$.)

(a) Determine whether A and B are similar. If so, find the matrix P such that $P^{-1}AP = B$. **(10 points.)**

(b) Compute $\exp A$. **(10 points.)**

4. Let v_1 and v_2 be two linearly independent vectors in $\mathbb{R}^2$. The set $\{mv_1 + nv_2 : m, n \in \mathbb{Z}\}$ is called the *lattice* spanned by v_1 and v_2. Now let L be the lattice spanned by $(1, 0)$ and $(0, 1)$ and M be the lattice spanned by $(4, 6)$ and $(2, 8)$. Find two vectors $v_1, v_2 \in L$ and two integers m_1 and m_2 such that v_1 and v_2 span L, while m_1v_1 and m_2v_2 span M. (**10 points.** *Hint*: Essentially you are asked to find the Smith normal form of a certain matrix.)