# 九十九學年度國立交通大學應用數學系博士班入學考試試題

科目：分析

Please explain *all* your answers and indicate which theorems you are using!

1. For $0 < \alpha \leq 1$, a function $f : [0,1] \longrightarrow \mathbb{R}$ is $\alpha$-Hölder continuous if there is a positive constant $C$ such that
$$|f(x) - f(y)| \leq C|x-y|^{\alpha}, \qquad \text{for all } x, y \in [0,1].$$
   (a) (7%) Prove that $g(x) = \sqrt{x}$ for $x \in [0,1]$ is $\frac{1}{2}$-Hölder continuous.
   (b) (7%) Prove that $g(x) = \sqrt{x}$ for $x \in [0,1]$ is *not* 1-Hölder continuous.

2. Suppose that $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ is a sequence of differentiable functions such that $M = \sup_{x\in\mathbb{R}, n\in\mathbb{N}} |f_n'(x)| < \infty$ and that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}$.
   Prove or disprove (by giving a counterexample) the following statements:
   (a) (7%) The sequence of functions $f_n$ is *uniformly bounded* for any fixed $[a,b] \subset \mathbb{R}$.
   (b) (7%) The function $f$ is *continuous* on $\mathbb{R}$.
   (c) (7%) The function $f$ is *differentiable* on $\mathbb{R}$.

3. (10%) Let $\Lambda$ be a nonempty (possibly uncountable) index set. For each $\lambda \in \Lambda$, $f_\lambda : \mathbb{R} \longrightarrow [0,1]$ is a continuous function. Define $g(x) = \sup_{\lambda\in\Lambda} f_\lambda(x)$. Prove that $g$ is Lebesgue measurable.

4. (a) (10%) Suppose $f : [0,1] \longrightarrow \mathbb{R}$ is an $L^1$-function with respect to the Lebesgue measure $m$ on $\mathbb{R}$. Compute $\displaystyle\lim_{n\to\infty} \int_{[0,1]} \left(\frac{nx^2}{1+nx}\right) \cdot f(x)\, dm(x)$.
   (b) (15%) Suppose $f : [0,1] \longrightarrow \mathbb{R}$ is an $L^p$-function with respect to the Lebesgue measure $m$ on $\mathbb{R}$, where $1 \leq p \leq \infty$. Define $g : [0,\infty] \longrightarrow [0,1]$ by $g(t) = m(\{x \in [0,1] : |f(x)| \geq t\})$. Find *all possible* $p$ such that $\displaystyle\int_{[0,\infty]} g(t)\, dm(t) < \infty$.

5. (10%) Let $f_n$ be a sequence of functions in $L^p$, $1 \leq p < \infty$, which converges almost everywhere to a function $f$ in $L^p$. Show that $f_n$ converges to $f$ in $L^p$ if and only if $\|f_n\|_p \to \|f\|_p$.

6. Let $\mu$ be a measure on a nonempty set $X$. For each $f \in L^\infty(X,\mu)$, define a multiplication operator $M_f$ on $L^2(X,\mu)$ into $L^2(X,\mu)$ by $M_f(g) = f \cdot g$ for all $g \in L^2(X,\mu)$.
   (a) (5%) Prove that $\|M_f\| = \sup\{M_f(g) : \|g\|_2 \leq 1\} \leq \|f\|_\infty$.
   (b) (15%) Find a necessary and sufficient condition for $f \in L^\infty(X,\mu)$ such that $M_f$ map $L^2(X,\mu)$ *onto* $L^2(X,\mu)$. Justify your answer!