Discrete Mathematics

Entrance Examination for Ph.D. Program

May 4, 2010

Carefully justify all your answers. Answers without explanation will not receive any score.

1. Assume that the numbers $1, 2, \ldots, n$ are placed in that order and clockwise on the circle. Start at 1 and clockwise delete every second number until all numbers are deleted. Denote the last number deleted by $J(n)$.

   (a) (6%) Show that for $n \geq 1$

   $$J(2n) = 2J(n) - 1, \qquad J(2n+1) = 2J(n) + 1$$

   with initial condition $J(1) = 1$.

   (b) (14%) Let $n = \sum_{k=0}^{m} a_k 2^k, a_m = 1$ be the binary expansion of n. Show that

   $$J(n) = \sum_{k=1}^{m} a_{k-1} 2^k + a_m.$$

2. (a) (15%) Let $a_1, a_2, \ldots, a_{17}$ be a sequence of integers. Show that there exists either an increasing subsequence of length 4 or a decreasing subsequence of length 4.

   (b) (5%) Find a generalization of part (a) for an arbitrary sequence $a_1, a_2, \ldots, a_n$ of integers.

3. (a) (4%) Let D_n denote the number of derangements of n elements $\{1, \ldots, n\}$. Show that the number of permutations of the set $\{1, \ldots, n\}$ with exactly t fixpoints is given by

   $$\binom{n}{t} D_{n-t}.$$

   (b) (8%) Let S be a set and denote by $P_1, \ldots, P_m$ properties which the elements in S have or not. Set

   $$A_i = \{x \in S : x \text{ has property } P_i\}.$$

Show the following generalization of the Inclusion-Exclusion principle: the number of elements from S which have exactly $t \leq m$ of the properties is given by

$$\sum |A_{i_1} \cap \cdots \cap A_{i_t}| - \binom{t+1}{t} \sum |A_{i_1} \cap \cdots \cap A_{i_{t+1}}| + \cdots + (-1)^{m-t} \binom{m}{t} |A_1 \cap \cdots \cap A_m|,$$

where the first sum runs over all t-combinations of $\{1, \ldots, m\}$, the second sum runs over all $t+1$-combinations of $\{1, \ldots, m\}$, etc.

(c) (8%) Use part (b) to give a different proof of part (a).

4. The Bernoulli numbers are defined as

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \frac{x^n}{n!}.$$

(a) (2%) Find B_0.

(b) (7%) Let

$$S_m(n) = \sum_{k=0}^{n-1} k^m.$$

Denote by $S_n(x)$ the exponential generating function of $S_m(n)$ (as a sequence of m). Find $S_n(x)$.

(c) (7%) Use part (b) to show that

$$S_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}.$$

(d) (4%) Follow from part (c) that

$$S_m(n) \sim \frac{n^{m+1}}{m+1},$$

where as usual $a_n \sim b_n$ if $\lim_{n \to \infty} a_n / b_n = 1$.

5. Consider a table with 2 rows and $n-1$ columns. Assume that the first row is $1, 2, \ldots, n-1$ and second row contains arbitrary numbers from the set $\{1, \ldots, n\}$. Construct an undirected graph G with vertices $\{1, \ldots, n\}$ by connecting the vertices in each column of the table.

(a) (4%) Show by an example that G is not necessarily a tree.

(b) (8%) If G is connected, show that it must be a tree.

(c) (8%) Show that every connected component of G contains at most one cycle.