Ph.D. entrance, 2010 — Linear Algebra

1. A sequence $\{a_n\}$ is defined recursively by the equations

$$a_0 = a_1 = 1, \quad a_{n+2} + a_{n+1} + a_n = 0, n = 0, 1, 2, \cdots . \quad (*)$$

Let $x_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$. Then (*) can be rewritten as $x_{n+1} = Ax_n$, here A is a 2×2 matrix.

   (a) (10 points) Find A^n.

   (b) (5 points) Find the explicit solution for (*).

   (c) (5 points) What can you say about the sequence $a_0 = c_0, a_1 = c_1, \ a_{n+2} + \alpha a_{n+1} + \beta a_n = 0, n = 0, 1, 2, \cdots, \ c_0, c_1, \alpha, \beta \in R$?

2. Let A and B be two 2×2 matrices.

   (a) (10 points) If $A^2 = B^2 = I$, $AB + BA = O$, prove that there exist a real nonsingular 2×2 matrix P with

$$PAP^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad PBP^{-1} = \begin{bmatrix} 0 & c \\ \frac{1}{c} & 0 \end{bmatrix},$$

   for some $c \neq 0$.

   (b) (10 points) Can you find A and B such that $A^2 = B^2 = O$, $AB + BA = I$?

3. Let V be an n-dimensional real vector space and let T be a linear transformation from V into V such that the range and null space of T are identical.

   (a) (5 points) Show that n is even.

   (b) (10 points) Does there exist an example of such a linear transformation T from a vector space V into itself ?

4. (20 points) Let V be a vector space with a bilinear form B. Assume that whenever $x, y \in V$ are so that $B(x, y) = 0$, then $B(y, x) = 0$. Show that B is symmetric or alternating.(Bilinear means B is linear in each of two variables. Symmetric means $B(x, y) = B(y, x)$, for all x and y in V. Alternating means $B(x, x) = 0$, for all x in V.)

5. Let V be an n-dimensional real inner product space with inner product $(,)$. If $\{u_1, u_2, \cdots, u_n\}$ is a basis for V, and T is a linear transformation from V into itself,define the trace of T by $\text{tr}(T) = \sum_{i,j=1}^{n} g^{ij}(Tu_i, u_j)$, where $g_{ij} = (u_i, u_j)$, and $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$.

   (a) (10 points) Show that this definition does not depend on the chosen basis u_i.

(b) (10 points) Show that $\mathrm{tr}(S \circ T) = \mathrm{tr}(T \circ S)$, for all linear transformations S and T from V into itself, where $S \circ T$ is the composition of S and T.

(c) (5 points) Show that it is impossible to find two linear transformations S and T from V into itself, such that $S \circ T(v) - T \circ S(v) = v$ for all $v \in V$.