國立交通大學應用數學研究所博士班資格考試

科目：代數

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1. Let G be a finite group of order n.

   (a) (10 %) Let p be the smallest prime number dividing n. Let H be a subgroup of index p. Show that H is normal in G.

   (b) (10 %) Assume that $n = 29645 = 5 \times 7^2 \times 11^2$. Suppose that G has only one 5-Sylow subgroup. Must G be an abelian group? You need to explain your answer.

2. Let $\mathbb{F}_q$ denote a finite field with q elements.

   (a) (10 %) Let $\mathbb{K}$ be a finite extension of $\mathbb{F}_q$ with $[\mathbb{K} : \mathbb{F}_q] = m$ and let $f(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_0 \in \mathbb{F}_q[x]$ be the minimal polynomial of $\alpha \in \mathbb{K}$ over $\mathbb{F}_q$. Prove that

   $$\mathbf{Tr}_{\mathbb{K}/\mathbb{F}_q}(\alpha) = -\left(\frac{m}{d}\right) b_{d-1} \quad \text{and}$$
   $$\mathbf{N}_{\mathbb{K}/\mathbb{F}_q}(\alpha) = (-1)^m\, b_0^{m/d},$$

   where $\mathbf{Tr}_{\mathbb{K}/\mathbb{F}_q}$ and $\mathbf{N}_{\mathbb{K}/\mathbb{F}_q}$ denote the trace and norm from $\mathbb{K}$ to $\mathbb{F}_q$.

   (b) (10 %) Prove

   $$x^{q^m} - x = \prod_{c \in \mathbb{F}_q} \left((\sum_{j=0}^{m-1} x^{q^j}) - c \right)$$

   for any positive integer m.

3. (10 %) Let $f(X) = v_0(X - t_1) \cdots (X - t_n)$. The discriminant of f is defined by

   $$D(f) = (-1)^{n(n-1)/2} v_0^{2n-2} \prod_{i \neq j} (t_i - t_j).$$

   Let $n \geq 2$ be an integer and let the polynomial $f(x) = x^n + ax + b$ where a, b are the coefficients of f. Compute the discriminant $D(f)$ of f in terms of a, b and n.

4. (15 %) Let Γ be a free abelian group of rank $n \geq 1$. Let Γ' be a subgroup of Γ which is of rank n also. Let $\{v_1, \ldots, v_n\}$ be a basis of Γ, and let $\{w_1, \ldots, w_n\}$ be a basis of Γ'. Write

   $$w_i = \sum a_{ij} v_j, \quad a_{ij} \in \mathbb{Z}.$$

   Show that the index $[\Gamma : \Gamma']$ is finite and is equal to the absolute value of the determinant of the matrix $A = (a_{ij})$.

5. Let $K = \mathbb{C}(t)$ where t is transcendental over the the complex numbers $\mathbb{C}$ and $\mathbb{C}(t)$ is the field of rational functions over $\mathbb{C}$. Let n be a positive integer.

   (a) (10 %) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + t \in K[x]$ where $a_{n-1}, \ldots, a_1 \in \mathbb{C}[t]$ are polynomials in t such that $a_{n-1}(0) = \ldots = a_1(0) = 0$. Prove or disprove that $f(x)$ is irreducible over K.

   (b) (10 %) Let $g(x) = x^n + t$. Compute the Galois group of $g(x)$ over K. (That is, the Galois group of the splitting field of $g(x)$ over K.)

6. (15 %) Let R be a principal ideal domain and let M be a free module of rank n over R. Let $w_1, \ldots, w_s \in M$ be given and put

$$E := \{\phi \in \mathrm{Hom}_R(M, R) \mid \phi(w_i) = 0, i = 1, \ldots, s\}$$

where $\mathrm{Hom}_R(M, R)$ denotes the the set of R-module homomorphisms from M to R with an R-module structure given by $(rf)(m) = rf(m)$ for all $r \in R$ and $m \in M$. Is it true that E is a free module over R? Explain your answer.