Qualifying Exam.: Ordinary Differential Equations (常微)

September, 2011 <共2頁>

Do five out of the following six problems. Each problem is 20 points. Show all your works to get full credits.

1. Consider the following predator-prey system

$$\begin{cases} x' = \gamma x\left(1 - \dfrac{x}{K}\right) - \alpha xy \\ y' = y(\beta x - d) \qquad\qquad \gamma, K, \alpha, \beta, d > 0 \\ x(0) > 0, \quad y(0) > 0. \end{cases} \tag{1}$$

Show that the solutions $x(t),\ y(t)$ of (1) are defined for all $t > 0$ and the solutions are positive and bounded for all $t > 0$.

2. Consider the IVP

$$\begin{cases} x' = Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x, \quad x(t) \in \mathbb{R}^2, \\ x(0) = x_0 \neq 0. \end{cases} \tag{2}$$

Assume that the real matrix A is nonsingular ($ad - bc \neq 0$). Give necessary and sufficient condition(s) on matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that the trajectory of IVP (2) is a circle on the phase plane. Give a mathematical proof.

3. (a) Solve $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$ of the linear system

$$x' = Ax = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} x. \tag{3}$$

where $\lambda \in \mathbb{R}$.

(b) Analyze and draw the phase portrait of linear system (3) in the $x_1x_2x_3$-space. (Discuss the cases (i) $\lambda > 0$, (ii) $\lambda = 0$, (iii) $\lambda < 0$.)

4. Consider nonlinear undamped pendulum equation $\theta'' + \sin\theta = 0$. Let $\omega = \theta'$, we obtain the equivalent first order nonlinear system

$$\begin{cases} \theta' = \omega \\ \omega' = -\sin\theta. \end{cases} \tag{4}$$

Problem. Draw the global phase portrait of the system on the (θ, ω)-plane and verify your results. In particular,

(a) Find and classify all equilibrium points (e.g. improper node, proper node, saddle point, spiral point, center, $\cdots$) of nonlinear system (4). (Give proofs of the results.)

(b) Find a Lyapunov function (energy function) for nonlinear system (4) and classify all the orbits by the Lyapunov function.

(c) Show the existence of saddle connections (heteroclinic orbits) of nonlinear system (4).

5. **Problem. Apply the following given Theorems 1 and 2 to prove Floquet's Theorem, parts (i) and (ii). You are not required to prove Theorems 1 and 2.**

Consider the nonautonomous linear system

$$x' = A(t)x, \tag{5}$$

where $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$ is a continuous, T-periodic matrix (i.e., $A(t) = A(t+T)$ for all t).

Theorem 1. Let $\Phi(t)$ and $\Psi(t)$ be two fundamental matrices of matrix equation

$$X' = A(t)X \tag{6}$$

where $X = (x_{ij}(t)) \in \mathbb{R}^{n\times n}$ and $X' = \left(x'_{ij}(t)\right)$. Then there exists a real, nonsingular matrix $\tilde{P}$ such that $\Psi(t) = \Phi(t)\tilde{P}$ for all t.

Theorem 2. Let $B \in \mathbb{R}^{n\times n}$ be nonsingular. Then there exists $A \in \mathbb{C}^{n\times n}$, called logharithm of B, satisfying $e^A = B$.

(Floquet's Theorem) (i) If $\Phi(t)$ is a fundamental matrix of (6), then so is $\Phi(t+T)$. Moreover there exists $P(t) \in \mathbb{C}^{n\times n}$ which is nonsingular, differentiable and satisfies $P(t) = P(t+T)$ for all t, and there exists $R \in \mathbb{C}^{n\times n}$ such that $\Phi(t) = P(t)e^{tR}$.

(ii) Under the hypotheses of part (i), the nonautonomous linear system (5), under the linear change of coordinates $y = P^{-1}(t)x$, reduces to the autonomous linear system $y' = Ry$.

6. Consider the system

$$\begin{cases} x' = -2y + yz \\ y' = x - xz \\ z' = xy \end{cases}$$

Problem. (a) Investigate the stability, asymptotic stability, and instability of the origin. State the theorem used.

(b) Is the Hartman-Grobman theorem applicable to study the stability, asymptotic stability, and instability of the origin? (Why?)