Department of Applied Mathematics
National Chiao Tung University
Ph.D. Qualifying Examination
Discrete Mathematics (離散)
September 2011

Problem 1.(15%) Let G be a finite connected graph without isthmuses. Show that G admits a strong orientation, i.e. an orientation that is a strongly connected digraph.

Problem 2.(15%) Prove that for any $k \geq 2$, there is $n > 3$ such that for any k-coloring of $\{1, 2, 3, \ldots, n\}$, there are three integers x, y, z (not necessarily distinct) of the same color such that $x + y = z$.

Problem 3.(15%) Let G be a simple graph with n vertices. Prove that if each vertex of G has degree $\geq (n+1)/2$, then for any edge e, there exists a Hamiltonian circuit of G that passes through e.

Problem 4.(15%) Color the integers 1 to $2n$ red or blue in such a way that if i is red then $i - 1$ is not blue. Use this to prove that

$$\sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} 2^{2n-2k} = 2n + 1.$$

Problem 5.(15%) Consider the set S of all ordered k-tuples $\mathcal{A} = (A_1, \ldots, A_k)$ of subsets of $\{1, 2, \ldots, n\}$. Show that

$$\sum_{\mathcal{A} \in S} |A_1 \cup A_2 \cup \cdots \cup A_k| = n(2^k - 1)2^{(n-1)k}.$$

Problem 6.(15%) Fix a finite field $\mathbb{F}_q$ and let N_d be the number of monic irreducible polynomials of degree d over $\mathbb{F}$. Show that

$$q^n = \sum_{d|n} dN_d.$$

Problem 7.(10%) Fibonacci numbers are defined by $F(0) = F(1) = 1$, $F(n) = F(n-1) + F(n-2)$. Prove that $F(n)$ is even if and only if $n \equiv 2 \pmod 3$.