PDE Qualifying Exam. (偏微)

September 2011

Do all the following five problems.

1. (18 points) Let $p = u_x$, $q = u_y$. Consider the first order nonlinear equation

$$u = p^2 - 3q^2, \quad x, t \in \mathbf{R}, \ t \geq 0,$$

with the initial condition $u(x, 0) = x^2$. Solve $u = u(x, y)$ by the method of characteristics.

2. (16 points) (a) Let u be a solution of

$$a(x, y)u_x + b(x, y)u_y = -u \qquad (a^2(x, y) + b^2(x, y) > 0)$$

of class C^1 in the closed unit disk Ω in the xy-plane. Let

$$a(x, y)x + b(x, y)y > 0$$

on the boundary of Ω. Prove that u vanishes identically.

(6 points) (b) Show that the result (u vanishes identically) in part (a) still holds if

$$a(x, y)x + b(x, y)y \geq 0$$

on the boundary of Ω.

3. Let Ω denote the unbounded set $|x| > 1$.

(10 points) (a) Let $u \in C^2(\overline{\Omega})$, $\Delta u = 0$ in Ω and $\lim_{x \to \infty} u(x) = 0$. Show that

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u| . \tag{1}$$

(6 points) (b) If the condition $\lim_{x \to \infty} u(x) = 0$ in part (a) is deleted, does the result in (1) still hold? Give a counterexample or a proof.

4. Let Ω be a domain in $\mathbf{R}^n$.

(12 points) (a) Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0$ in Ω. For any ball $B = B_R(y) \subset\subset \Omega$, show that

$$u(y) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B} u ds \ \ \text{(sphere mean)}$$

and

$$u(y) = \frac{n}{\omega_n R^n} \int_B u dx \ \ \text{(solid ball mean)},$$

where ω_n is the surface area of the unit sphere in $\mathbf{R}^n$. (Note that $\omega_n = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.)

(8 points) (b) Suppose $u \in C^2(\Omega)$ satisfies

$$u(x) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x)} u ds \ \ \text{(sphere mean)}$$

for all ball $B_R(x) \subset\subset \Omega$. Show that u satisfies $\Delta u = 0$ in Ω.

5. Consider the one-dimensional nonhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = \varphi(x, t) \in C^1, & x \in \mathbf{R}, t \in \mathbf{R}^+, \\ u(x,0) = f(x) \in C^2, & x \in \mathbf{R}, \\ u_t(x,0) = g(x) \in C^1, & x \in \mathbf{R}. \end{cases} \tag{2}$$

(10 points) (a) Write and derive the d'Alembert solution of (2).

(2 points) (b) Find the domain of dependence of the point (x_0, t_0) with $x_0 \in \mathbf{R}$ and $t_0 > 0$.

(12 points) (c) Show that there is exactly one C^2-solution of (2).