博士班資格考

Probability, SEP 2011 (機率)

20 points for each problem
Convergence here is for $n \to \infty$.

1)**Kolmogorov's inequality** Let $X_1, \cdots, X_n$ be independent with $EX_k = 0$ and $var(X_k) < \infty$, for $1 \le k \le n$. Let $S_n = X_1 + \cdots + X_n$, show that for any $x > 0$,

$$P(\max_{1\le k\le n} \mid S_k \mid \ge x) \le \frac{var(S_n)}{x^2}.$$

2)Let $X_1, \cdots, X_n, \cdots$ and X be random variables defined in the same probability space.
i)If X_n converges to X in probability, show that X_n converges to X weakly.
ii)If the random vector (X_n, X) converges to (X, X) weakly, show that X_n converges to X in probability.

3)**General Bayes formula** Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ on which two probability measures Q and P are given. If $Q << P$ with Radon-Nikodym derivative Λ and X is Q-integrable, show that $X\Lambda$ is P-integrable and

$$E_Q(X \mid \mathcal{G}) = \frac{E_P(X\Lambda \mid \mathcal{G})}{E_P(\Lambda \mid \mathcal{G})}, \quad Q - a.s.$$

4)Let $X_1, \cdots, X_n, \cdots$ be *i.i.d.* with $Ee^{X_1} < \infty$. Let $\mathcal{F}_n$ denote the σ-algebra generated by $X_1, \cdots, X_n$. Put $M_n = e^{\left(\sum_{k=1}^n X_k\right) - nh}$, where $h = lnEe^{X_1}$. Show that
i)M_n is a martingale w.r.t. $\{\mathcal{F}_n\}$.
ii)If X_1 is not a constant, then $M_n \to 0 \ \ a.s.$

5)Let $X_0, X_1, \cdots, X_n, \cdots$ be a Markov chain with a countable state space. Show that for $0 < k < n$, we have

$$P\{X_k = i_k \mid X_l = i_l, l \ne k, 0 \le l \le n\} = P\{X_k = i_k \mid X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\}.$$