Qualifying Exam.: Ordinary Differential Equations **February 24 2012**

This exam. contains 5 problems with a total of 100 points. Do all 5 problems.

1. (15 points) Let

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

be a C^1-autonomous system on the plane (i.e. $f, g \in C^1$). Show that the solutions exist for all real time t if $f^2 + g^2 \leq 100$ on the plane.

2. (15 points) (a) Let $n = 2$. For any 2×2 constant real matrix A, show that there exists an invertible real matrix P such that the matrix

$$B = P^{-1}AP$$

has one of the following forms

$$\textbf{(i)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad \textbf{(ii)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \textbf{(iii)} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where $\lambda,\mu,a,b \in \mathbb{R}$. **Find P explicitly.**

(10 points) (b) Let

$$A = \begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix},$$

where $\lambda, \mu, \alpha \in \mathbb{R}$. Solve the IVP

$$x' = Ax, \ \ x(0) = x_0.$$

3. (15 points) Find a suitable real 4×4 matrix A such that each nontrivial solution of $x' = Ax$, $x \in \mathbb{R}^4$ is **bounded** for $t \in (-\infty, \infty)$ and **nonperiodic**. Justify your answer.

4. (15 points) Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -x \\ y + x^2 \end{bmatrix}$$

and let $A = DF(0)$. Show that the flows generated by the differential equations $z' = F(z)$ and $z' = Az$ are topologically conjugate. (Hint: Try **quadratic** conjugacy functions for a homeomorphism $H = (\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix})$.)

5. (15 points) (a) Let 0 be an isolated equilibrium point of the system

$$x' = f(x) \in C^1.$$

Let $V(x)$ be a positive and continuously differentiable function defined on a neighborhood D of the origin, satisfying $V(0) = 0$. Assume that in any subset, containing the origin, of D, there is an $\tilde{x}$ such that $V(\tilde{x}) > 0$. If, moreover

$$\dot{V}(x) \equiv \operatorname{grad} V(x) \cdot f(x) > 0 \ \text{ for all } x \neq 0 \text{ in } D.$$

Show that the equilibrium point 0 is **unstable**.

(15 points) (b) Apply part (a) to show that the zero solution $x(t) \equiv 0$ is an **unstable** solution of the equation $x'' - x + x' \sin x = 0$. (Hint: Try **quadratic** functions for $V(x, y)$.)