National Chiao Tung University
Department of Applied Mathematics
Discrete Mathematics Qualifying Examination
February 2012

Problem 1.(10pts) Prove that a finite graph G with no isolated vertices (but possibly with multiple edges) is Eulerian if and only if it is connected and every vertex has even degree.

Problem 2.(10pts) Prove the following result by using counting method: There are n^{n-2} different labeled trees on n vertices.

Problem 3.(10pts) For a given n, prove that there is an integer $N(n)$ such that any collection of $N \geq N(n)$ points in the plane, no three on a line, has a subset of n points forming a convex n-gon.

Problem 4.(10pts) Suppose that G is a simple graph with n vertices and has all vertices of degree at least $n/2$. Prove that G has a Hamiltonian circuit.

Problem 5.(10pts) Let $A = (a_{ij})$ be an $n \times n$ matrix with nonnegative integers as entries, such that every row and column of A has sum ℓ. Prove that A is the sum of ℓ permutation matrices.

Problem 6.(10pts) Let a_n denote the number of paths of length n in the X-Y plane starting from $(0,0)$ with steps $R : (x,y) \to (x+1,y)$, $L : (x,y) \to (x-1,y)$, and $U : (x,y) \to (x,y+1)$) such that a step R is not followed by a step L and vice versa. Find $\lim_{n\to\infty} a_n^{1/n}$. Prove your answer.

Problem 7.(10pts) For a 2-(v,k,λ) design with b blocks and $v > k$, prove that $b \geq v$.

Problem 8.(10pts) Let G be a connected planar graph with N vertices $v_1, v_2, \ldots, v_N$. For $i = 1, 2, \ldots, N$ let S_i be a set of five elements (which we call colors). Prove that there exists a mapping f on the vertices of G, with $f(v_t) \in S_t$ $(t = 1, 2, \ldots, N)$. such that $f(v_i) \neq f(v_j)$ for all adjacent pairs v_i, v_j.

Problem 9.(20pts) Given a set $S = \{1, 2, \ldots, n\}$, find how many subsets of S have cardinality divisible by 3? Prove your answer.