博士班資格考 103年9月

PhD Qualifying Exam in Numerical Analysis

Fall 2014

1. (20%) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix and $\boldsymbol{b} \in \mathbb{R}^n$.
   (a) Solving the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is equivalent to minimizing a quadratic functional $\Phi : \mathbb{R}^n \to \mathbb{R}$. Please define the functional Φ and then prove the equivalence.
   (b) Use the quadratic functional Φ in (a) to derive the gradient method with optimal step size for solving $\boldsymbol{Ax} = \boldsymbol{b}$.

2. (20%) Let f be a given real-valued function in $C^{n+1}[a,b]$. Let $x_0, x_1, \cdots, x_n$ be $n+1$ distinct real numbers in the interval $[a,b]$ and $y_0 = f(x_0), y_1 = f(x_1), \cdots, y_n = f(x_n)$.
   (a) Prove that there exists a unique polynomial Π_n of degree at most n such that
   $$\Pi_n(x_i) = y_i \quad \text{for } i = 0, 1, \cdots, n.$$
   (b) Prove that for each x in $[a,b]$ there corresponds a point ξ_x in (a,b) such that
   $$f(x) - \Pi_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i).$$

3. (20%) Let f be sufficiently smooth and satisfy the Lipschitz condition such that there exists a unique solution $y(t)$ for $t_0 \le t \le t_0 + T$ of the following initial value problem (IVP):
   $$\begin{cases} y'(t) &= f(t, y(t)) \quad \text{for } t_0 < t < t_0 + T, \\ y(t_0) &= y_0 \in \mathbb{R}. \end{cases}$$
   Consider the $(p+1)$-step method for approximating the IVP:
   $$u_{n+1} = \sum_{j=0}^{p} a_j u_{n-j} + h \sum_{j=0}^{p} b_j f_{n-j} + h b_{-1} f_{n+1}, \quad n = p, p+1, \cdots,$$
   where $p \ge 0$, u_j is the approximation to $y(t_j)$ and f_j denotes the value $f(t_j, u_j)$.
   (a) Please explain the following terminologies for the multistep method: convergence, zero-stability, consistence condition, and root condition. What are the relationship among the above concepts?
   (b) Consider the explicit 2-step method,
   $$u_{n+1} = 3u_n - 2u_{n-1} + \frac{h}{2}\Big(f_n - 3f_{n-1}\Big), \quad n \ge 1.$$
   Is the method suitable for solving the IVP? Why?

4. (20%) Let $\Omega \subseteq \mathbb{R}^2$ be an open bounded domain with a smooth boundary $\partial\Omega$ and $f \in L^2(\Omega)$. Consider the following boundary value problem (BVP) for the reaction-convection-diffusion equation:

$$\begin{cases} -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon > 0$ is the viscosity coefficient and $\boldsymbol{a} = (a_1, a_2)^\top$ is the constant velocity vector.

(a) Derive the weak formulation of the BVP in a suitable Sobolev space.

(b) Let V be a real Hilbert space with scalar product $(\cdot,\cdot)_V$ and associated norm $\|\cdot\|_V$. Let $B : V \times V \to R$ be a bilinear form and $L : V \to \mathbb{R}$ be a linear form. State the Lax-Milgram lemma.

(c) Use (b) to prove that the weak problem in (a) has a unique solution.

(d) Prove that the continuous piecewise linear finite element solution u_h of the BVP satisfies the following error estimate provided $u \in H^2(\Omega)$:

$$\|u - u_h\|_{H^1(\Omega)} \le Ch^1\|u\|_{H^2(\Omega)}.$$

5. (20%) Let us consider the following scalar hyperbolic problem:

$$\begin{cases} u_t + au_x &= 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

where $a > 0$ is a constant. Let Δt be the time step and Δx the spatial grid size.

(a) Derive the Lax-Friedrichs method directly for solving the above scalar hyperbolic problem, and then show that the method looks like a discretization of following advection-diffusion equation using the forward difference in time and centered difference in space:

$$u_t + au_x = \nu u_{xx},$$

where $\nu := (\Delta x)^2/2\Delta t$.

(b) What is the Courant-Friedrichs-Lewy (CFL) condition for the Lax-Friedrichs method? Show that the Lax-Friedrichs method is stable under the CFL condition in the discrete norm $\|\cdot\|_{\Delta,1}$ given by

$$\|\boldsymbol{v}\|_{\Delta,1} := \Delta x \sum_{j=-\infty}^{\infty} |v_j|.$$