PhD Qualify Exam in Numerical Analysis

Spring 2017

(1) Consider the special model problem

$$\begin{cases} -u''(x) + u(x) = (1+\pi^2)\sin \pi x, \text{ for } 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

which exact solution is $u(x) = \sin(\pi x)$. Using the centered difference method with uniform mesh to discretize (1), it results a linear system with matrix form

$$A\mathbf{u}_h = \mathbf{b}, \quad A \in \mathbb{R}^{n \times n}, \tag{2}$$

where n is the mesh number.

(a) (15%) Show that the linear system (2) has exact solution $\mathbf{u}_h^* = \left[1 + 4h^{-2}\sin^2\left(\frac{\pi h}{2}\right)\right]^{-1}\mathbf{b}$ and $\|\mathbf{u} - \mathbf{u}_h^*\|_\infty \le \frac{\pi^4}{12}h^2$, where $h = (n+1)^{-1}$ and $\mathbf{u} = [\sin(\pi h), \sin(2\pi h), \cdots, \sin(n\pi h)]^\top$. (Hint $\cos(2\theta) = 1 - 2\sin^2\theta$ and $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$)

(b) (15%) Show that the coefficient matrix A in (2) is not only strictly diagonal dominant but also symmetric positive definite.

(c) (15%) Prove that for any initial vector, both the Jacobi and Gauss-Seidel methods converge to the unique solution of (2). (Please DO NOT directly use the theoretical results, e.g., if A is strictly diagonal dominant, then both the Jacobi and Gauss-Seidel methods converge for any initial vector.)

(d) (10%) For a general matrix A, the Gauss-Seidel method is always superior to the Jacobi method or not. For the linear system (2), which one is superior? Why?

(e) (10%) What are the differences between the gradient and conjugate gradient methods in solving linear system with symmetric positive definite coefficient matrix?

(2) (15%) Let $x^* \in [1, 2]$ be the solution of

$$x^3 + 4x^2 - 10 = 0.$$

Define sequences $\{x_k\}$ and $\{y_k\}$ as

$$x_k = \frac{1}{2}\left(10 - x_{k-1}^3\right)^{1/2} \quad \text{for } k = 1, 2, \ldots$$

with any $x_0 \in [1, 1.5]$ and

$$y_k = \sqrt{10/(4 + y_{k-1})} \quad \text{for } k = 1, 2, \ldots$$

with any $y_0 \in [1, 2]$. Prove that $x_k \to x^*$ and $y_k \to x^*$ as $k \to \infty$. Which one will converge fast? ($\frac{27}{8}\sqrt{\frac{2}{53}} \approx 0.66$, $\frac{2}{7\sqrt{3}} \approx 0.17$ and $\frac{1}{5\sqrt{2}} \approx 0.14$)

(3) (10%) Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ that uses $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$, $f(x_0+2h)$, and $f(x_0+3h)$.

(4) (10%) Define the degree of precision of a quadrature formula

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

to be the largest positive integer n such that the formula is exact for x^k, for each $k = 0, 1, \ldots, n$. Please prove that the degree of precision of the quadrature formula is n if and only if the error of the approximation is zero for all polynomials of degree $k = 0, 1, \ldots, n$ but is not zero for some polynomial of degree $n+1$.