NCTU Department of Applied Mathematics
Qualifying Examination in Analysis for the Ph.D. Program Sep. 2017

1. (a)(10%) Assume that f is a real function on $\mathbb{R}$ and satisfies

$$f(\int_0^1 \phi(x)dx) \le \int_0^1 f(\phi)dx$$

for every measurable bounded function ϕ. Must f be a convex function?
(b)(10%) Let ν be a positive measure on Ω and $f : \Omega \to (0, \infty)$ with $\int_\Omega f d\nu = 1$. Prove that for every measurable subset $\omega \subset \Omega$ with $0 < \nu(\omega) < \infty$, that

$$\int_\omega \log f d\nu \le \nu(\omega) \log(\nu(\omega)^{-1}).$$

2. (20%) Let $(S, \mathcal{S}, \mu)$ be a measure space. Show that if $\mu(S) < \infty$, $g_n \to g$ in measure and $h_n \to h$ in measure, then $g_n h_n \to gh$ in measure. Does the statement remain true if the finiteness condition of $\mu(S)$ is removed?

3. (a)(10%) Let $f, f_1, f_2, \cdots$ be complex-valued measurable functions on $(S, \mathcal{S}, \mu)$. We say that $f_n \to f$ *almost uniformly* if for any $\varepsilon > 0$, there exists a set $B \in \mathcal{S}$ such that $\mu(B) < \varepsilon$ and $f_n \to f$ uniformly on B^c. Show that if $f_n \to f$ almost uniformly, then $f_n \to f$ almost everywhere and $f_n \to f$ in measure.
(b)(15%) Does the converse to (a) hold?

4. Prove or disprove the statements.
(a)(10%) $|f|$ is measurable $\Rightarrow$ f is measurable.
(b)(15%) Let f be a real-valued function on $\mathbb{R}$. Then the set of discontinuous points of f is an F_σ set (a countable union of closed sets). (Hint: it suffices to show that the set of continuous points is a G_δ set, a countable intersection of open sets.)

5.(10%) Justify rigorously the following limit

$$lim_{n\to\infty} \int_0^\infty (1 + \frac{x}{n})^{-n} sin\frac{x}{n} dx = 0.$$