PhD Qualifying Exam in Numerical Analysis

Spring, 2018

(Total of 110 points.)

1. Consider the problem $-u''(x) = 2$ for all $x \in (0, 1)$, $u'(0) = 0$, $u(1) = 0$.

   (A) (10 pts) Find an exact solution $u(x)$ of the problem, if it exists.

   (B) (10 pts) Use the central finite difference method to find a piecewise linear approximate solution $U(x)$ of the problem with a uniform mesh of 5 grid points. Draw $U(x)$ and compare it with $u(x)$.

   (C) (15 pts) What is the convergence order of your finite difference method? Show your proof. (Hint: Taylor series.)

2. The following conjugate gradient (CG) method is an algorithm for solving $A\vec{x} = \vec{b}$, where the matrix $A \in R^{N \times N}$ is symmetric and positive definite.

$\vec{x}^{(0)}$ (arbitrary), $\vec{r}^{(0)} = \vec{b} - A\vec{x}^{(0)}$, $\vec{p}^{(1)} = \vec{r}^{(0)}$

for $k = 1, \cdots, N$

(S1) $\alpha_k = \dfrac{\langle \vec{p}^{(k)}, \vec{r}^{(k-1)} \rangle}{\langle \vec{p}^{(k)}, A\vec{p}^{(k)} \rangle}$

(S2) $\vec{x}^{(k)} = \vec{x}^{(k-1)} + \alpha_k \vec{p}^{(k)}$

(S3) $\vec{r}^{(k)} = \vec{b} - A\vec{x}^{(k)} = \vec{r}^{(k-1)} - \alpha_k A\vec{p}^{(k)}$

check convergence; continue if necessary

(S4) $\beta_k = \dfrac{-\langle \vec{r}^{(k)}, A\vec{p}^{(k)} \rangle}{\langle \vec{p}^{(k)}, A\vec{p}^{(k)} \rangle}$

(S5) $\vec{p}^{(k+1)} = \vec{r}^{(k)} + \beta_k \vec{p}^{(k)}$

end for

   (A) (10 pts) Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find the solution of $A\vec{x} = \vec{b}$ using the CG method with the initial guess $\vec{x}^{(0)} = \begin{bmatrix} 1 \\ 1/9 \end{bmatrix}$.

   (B) (10 pts) Show that the set $\{\vec{p}^{(1)}, \vec{p}^{(2)}, \vec{p}^{(3)}, \cdots, \vec{p}^{(N)}\}$ obtained by the CG method is A-orthogonal. (Hint: (S5))

(C) (15 pts) Show that $A\vec{x}^{(N)} = \vec{b}$, i.e., CG finds the exact solution $\vec{x}^{(N)}$ in N steps. (Hints: (S2), $A\vec{x}^{(N)} = A\vec{x}^{(N-1)} + \alpha_N A\vec{p}^{(N)}$, $\left\langle A\vec{x}^{(N)} - \vec{b}, \vec{p}^{(k)} \right\rangle$)

(D) (10 pts) Let $\phi(\vec{x}) = \frac{1}{2}\langle A\vec{x}, \vec{x}\rangle - \left\langle \vec{b}, \vec{x} \right\rangle$ and $h(\alpha) = \phi(\vec{x} + \alpha\vec{p})$ for all $\vec{x} \in R^N$ and a given $\vec{p}$. Show that $\alpha^* = \dfrac{\langle \vec{p}, \vec{r}\rangle}{\langle \vec{p}, A\vec{p}\rangle}$ minimizes $h(\alpha)$.

3. Newton's method finds successively approximations to a root (*unknown solution*) x^* of a nonlinear equation

$$g(x) = 0, \tag{3.1}$$

i.e., it iteratively solves the linearized equation

$$\begin{aligned} g'(x^{(0)})w &= g(x^{(0)}), \quad w = x^{(0)} - x^{(1)}, && (3.2) \\ g'(x^{(0)})w &= \lim_{t \to 0} \frac{g(x^{(0)} + tw) - g(x^{(0)})}{t}, && (3.3) \end{aligned}$$

where $x^{(1)}$ is the next iterate (unknown) to be solved with a given $x^{(0)}$, then $x^{(2)}$ is solved with $x^{(1)}$, and so on.

(A) (15 pts) For the coupled nonlinear system

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 = f_1(x_1, x_2) \\ a_{21}x_1 + a_{22}x_2 = f_2(x_1, x_2) \end{array} \right. \tag{3.4}$$

written in the matrix form $AX = F$ with two *unknown solutions* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X$, the linear operator $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (a matrix), and two nonlinear functions $\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = F$, derive the linearized system of (3.4) in matrix form that corresponds to (3.2) and (3.3).

(B) (15 pts) For the semilinear (nonlinear) differential equation (DE)

$$-u''(x) = f(u(x)) = e^{u(x)} \tag{3.5}$$

with an *unknown solution* $u(x)$, the positive linear operator $-\frac{d^2}{dx^2}$, and the nonlinear functional $f(u)$, derive the linearized DE of (3.5) that corresponds to (3.2) and (3.3). Show that $-\frac{d^2}{dx^2}$ is linear.