92學年度博士班資格考－實變分析

P1

Real Analysis

92.9月

(13%) 1. Let $(X, \mathcal{M}, \mu)$ be a positive measure space, $f \in L^1(\mu)$, S a closed set of the complex plane, and the averages

$$\frac{1}{\mu(E)} \int_E f \, d\mu \in S$$

for $E \in \mathcal{M}$ and $\mu(E) > 0$. Prove that $f(x) \in S$ for almost all $x \in X$.

(13%) 2. Prove that each Lebesgue measurable subset of $\mathbb{R}^n$ can be expressed as a union of an F_σ-set and a Lebesgue measure zero set.

(13%) 3. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lebesgue integrable, and λ the Lebesgue measure on $\mathbb{R}^n$. Let

$$A_k = \{x \in \mathbb{R}^n \; ; \; |f(x)| > k\} \quad (k = 1, 2, \ldots).$$

(a) Show that

$$\lim_{k \to \infty} \int_{A_k} |f| \, d\lambda = 0.$$

(b) Show that corresponding to each $\varepsilon > 0$ there exists a bounded Lebesgue integrable function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} |g - f| \, d\lambda < \varepsilon.$$

(13%) 4. Find a representation for the bounded linear functionals on l^p, $1 \leq p < \infty$. Justify your answer.

(13%) 5. (a) Prove that $L^p(\mu)$ is a Banach space, for every $1 \leq p \leq \infty$ and for every positive measure μ.

(b) Prove that if $p \neq 2$, $L^p(\mu)$ is *not* a Hilbert space.

P2

(13%) 6. Let $X = [0,1]$, and λ be the Lebesgue measure. Define a sequence $\{f_n\}$ in $L^2(\lambda)$ by

$$f_n(t) = \begin{cases} \sqrt{t}, & \text{if } 0 \le t \le \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < t \le 1 \end{cases}.$$

Prove that $\{f_n\}$ does *not* converge to 0 with respect to $\|\cdot\|_2$, but for any bounded linear functional φ on $L^2(\lambda)$, $\{\varphi(f_n)\}$ converges to 0.

(13%) 7. Suppose $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0,1]$ with

$$\int_0^1 |f_n(x)|^2 \ dx \le 0 \quad (n = 1, 2, \ldots)$$

and $f_n \longrightarrow 0$ a.e. on $[0,1]$. Prove that

$$\int_0^1 |f_n| \ dx \longrightarrow 0.$$

(Hint : Use Egorov's theorem and Cauchy-Schwarz's inequality.)

(13%) 8. Let $C[0,1]$ be the vector space of all continuous complex-valued functions on $[0,1]$. For $f \in C[0,1]$, define

$$\|f\| = \max_{x \in [0,1]} |f(x)|.$$

(a) Prove that $(C[0,1], \|\cdot\|)$ becomes a Banach space.

(b) Prove that the closed unit ball $\{f \in C[0,1] \ ; \ \|f\| \le 1\}$ is *not* compact.

(c) Describe all compact sets in $C[0,1]$. Justify your answer.