(a) Show that there does not exist a μ-measurable function f such that

$$\lambda(E) = \int_E f\,d\mu \quad \text{for all } E \in \Sigma.$$

(b) State the Radon-Nikodym theorem and explain why (a) is not a counter-example.

7(12%). Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Prove that

$$\lim_{\|y\| \to 0} \|f(x+y) - f(x)\|_{L^p} = 0,$$

where the L^p-norm is taken with respect to the x-variable.

8(12%). Let X be a linear space which is complete in each of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, and suppose there is a constant c such that

$$\|x\|_1 \le c\,\|x\|_2 \quad \text{for all } x \in X.$$

Show that the norms are equivalent, i.e., there is a second constant m such that

$$\|x\|_2 \le m\,\|x\|_1 \quad \text{for all } x \in X.$$

9(12%). Let (X, μ) be a measure space in which $\mu(X) < \infty$, and let $\{f_n\}$ be an orthonormal sequence in $L^2(X, \mu)$. Suppose that there is a constant M such that $|f_n(x)| \le M$ a.e. for all n, and that $\sum_{n=1}^{\infty} a_n f_n(x)$ converges a.e. Prove that $\lim_{n\to\infty} a_n = 0$.

Ph.D. Qualifying Examination: Real Analysis

1(10%). Compute the Lebesgue integral of the function

$$f(x,y) = \begin{cases} 1 & \text{if } xy \text{ is irrational} \\ 0 & \text{if } xy \text{ is rational} \end{cases}$$

over the square $0 \le x \le 1$, $0 \le y \le 1$.

2(10%). Show that there is no function f such that both $f \in L^2(\mathbb{R})$ and $1/f \in L^2(\mathbb{R})$.

3(10%). Let $C[0,1]$ be the space of all continuous real-valued functions on $[0,1]$ with norm $\|f\| = \sup_{0\le x\le 1} |f(x)|$. Show that the set of all functions on $[0,1]$ of the form

$$g(x) = \sum_{j=1}^{n} a_j e^{b_j x}, \quad \text{where } a_j,\ b_j \in \mathbb{R},$$

is dense in $C[0,1]$.

4(10%). Compute the value of

$$\sup \left| \int_0^{2\pi} f(x) \cos x \, dx \right|$$

where the supremum is taken over all $f \in L^2[0, 2\pi]$ with $\|f\|_2 \le 1$.

5(12%). Let f be a mapping from the measurable space $(X, \mathcal{B})$ to the topological space Y.
(a) Prove that $\Omega = \{E \subset Y : f^{-1}(E) \in \mathcal{B}\}$ is a σ-algebra in Y.
(b) If f is measurable, prove that $f^{-1}(E) \in \mathcal{B}$ for every Borel set E in Y.

6(12%). Let $X = [0,1]$, and let Σ denote the σ-algebra of all Lebesgue measurable subsets of X. Let λ be Lebesgue measure on Σ, and let μ be the counting measure on Σ.