博士班資格考 - ODE

Qualifying Exam. Ordinary Differential Equations
Spring 2005

P4.3.

Do all six problems.

1. (15%) Consider $x' = Ax$, where A is a 2×2 real constant nonsingular matrix. Suppose that $x = \varphi(t)$ is a nontrivial real periodic solution. Show that the trajectory of $\varphi(t)$ lies on an ellipse on the phase plane.

2. (15%) Use the method of successive approximation to show that if the matrix valued function $A(t)$ is continuous on $[-a_0, a_0]$ for some $a_0 > 0$, then there exists $a > 0$ such that the initial value problem

$$\begin{cases} \dot{\Phi}(t) = A(t)\Phi \\ \Phi(0) = I \end{cases}$$

has a unique fundamental matrix solution $\Phi(t)$ on $[-a, a]$.

3. (15%) Let $(t, x) \in [0, \infty) \times \mathbf{R}^n$ $(n \geq 1)$ and $f(t, x) \in \mathbf{R}^n$ be continuously differentiable with

$$|f(t,x)| \leq g(t)h(|x|)$$

where g is continuous on $[0, \infty)$, $g(t) \geq 0$ for all $t \geq 0$, h is continuous on $[0, \infty)$, $h(u) \geq 1$ for all $u \geq 0$, and

$$\int_0^\infty \frac{du}{h(u)} = \infty.$$

Prove that for every ξ in $\mathbf{R}^n$ the initial value problem $x'(t) = f(t, x)$, $x(0) = \xi$ has a solution that exists for all $t \geq 0$.

4. (15%) Let $H : \mathbf{R}^2 \to \mathbf{R}$ via $(x, y) \to H(x, y)$ be given. Consider the system

$$\begin{cases} \dot{x} = \dfrac{\partial}{\partial y} H(x,y) \\ \dot{y} = -\dfrac{\partial}{\partial x} H(x,y). \end{cases}$$

(a) Let $(x(t), y(t))$ be a solution. Show that $h(t) = H(x(t), y(t))$ is constant.

(b) Let $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{40}x^5$.

(i) Find the equilibrium of the system.

(ii) Linearize about the equilibrium and discuss linear stability. What can you conclude about nonlinear stability. (State the standard theorem.)

(iii) Sketch the phase plane portrait (level curves of H) and complete the discussion of nonlinear stability.

(iv) What initial conditions give rise to periodic solutions. Discuss their stability in terms of Lyapunov and orbital stability.