博士班資格考－偏微

PDE Qualifying Exam (Feb. 2005)

This exam contains 7 problems with a total of 120 points. In doing each of the following problems, you can quote any known results appeared in the book by Fritz John, "Partial Differential Equations". In case you want to quote a standard theorem, state the name (if any) of that theorem.

To receive more points, your arguments should be as clear and explicit as possible. To pass the exam, your total points must be at least 60.

1. (20 points) (**equipartition of energy of wave equation**) Let $u(x,t) \in C^\infty(\mathbb{R}\times[0,\infty))$ solve the initial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R} \end{cases}$$

where $f(x)$ and $g(x)$ are given smooth functions which have compact support in $\mathbb{R}$. Define the **kinetic energy** as $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ and the **potential energy** as $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$. Show that

   (a) (3 points) $u(x,t)$ also have compact support in x for each fixed $t \in (0,\infty)$.

   (b) (7 points) The sum $k(t) + p(t)$ is a constant independent of $t \in (0,\infty)$.

   (c) (10 points) $k(t) = p(t)$ if t is large enough.

2. (10 points) Solve (i.e., find $u(x,y,z)$) the following first order quasi-linear equation in three variables $x,\ y,\ z$ with initial condition

$$\begin{cases} xu_x + yu_y + u_z = u \\ u(x,y,0) = h(x,y), \quad x,\ y \in \mathbb{R} \end{cases}$$

where $h(x,y)$ is a given smooth function.

3. (30 points) Assume u is harmonic in the domain Ω and let $B = B_R(y) \subset\subset \Omega$ be the ball centered at y with radius $R > 0$, which is strictly contained in Ω. Let $Du \in \mathbb{R}^n$ denote the **gradient** vector of u and ∂B denote the boundary of the ball B.

   (a) (10 points) Use **mean value formula** to show that we have

$$Du(y) = \frac{1}{\omega_n R^n}\int_{\partial B} u\nu dS$$

   where ν is the unit outward normal vector to ∂B, ω_n is the volume of the unit ball in $\mathbb{R}^n$, and the right hand side of the above equation means the surface integral of the vector function $u\nu$ on ∂B.

(b) (10 points) Use part (a) to show that we have the derivative estimate for harmonic function u on Ω :

$$|Du(y)| \leq \frac{n}{R} \sup_{\partial B} |u|, \quad B = B_R(y) \subset\subset \Omega.$$

(c) (10 points) If in addition that $u \geq 0$ on Ω, show that we have

$$|Du(y)| \leq \frac{n}{R} \cdot u(y), \quad B = B_R(y) \subset\subset \Omega.$$

4. (30 points) Consider the Cauchy problem for the heat equation in $\mathbb{R}^n$:

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{on} \quad x \in \mathbb{R}^n \end{cases}$$

where $f(x)$ is a given continuous and bounded function on $\mathbb{R}^n$. It is known that $u(x,t)$ given by

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy, \quad x \in \mathbb{R}^n, \quad t > 0$$

is a solution to the above Cauchy problem. In the following, assume u is given by the above formula and f is at least continuous and bounded.

(a) (5 points) Show that for each $t > 0$, we have the conservation law

$$\int_{\mathbb{R}^n} u(x,t)\, dx = \int_{\mathbb{R}^n} f(y)\, dy.$$

(b) (10 points) If we also assume that $f \in L^1(\mathbb{R}^n)$, show that

$$\lim_{t\to\infty} u(x,t) = 0 \quad \text{uniformly in} \quad x \in \mathbb{R}^n.$$

Give an example to show that if $f \notin L^1(\mathbb{R}^n)$, then the above estimate fails.

(c) (10 points) If we also assume that $\lim_{|x|\to\infty} f(x) = 0$, show that

$$\lim_{|x|\to\infty} u(x,t) = 0 \quad \text{for each fixed} \quad t > 0.$$

(d) (5 points) If we also assume that $f(x)$ is supported on the compact set $K \subset \mathbb{R}^n$ with $|f(x)| \leq C$ for all $x \in \mathbb{R}^n$, for some constant $C > 0$. Show that

$$|u(x,t)| \leq \frac{C\,|K|}{(4\pi t)^{n/2}}, \quad \text{for all} \quad x \in \mathbb{R}^n, \quad t > 0$$

where $|K|$ is the measure of K in $\mathbb{R}^n$.

5. (10 points) The maximum principle for harmonic functions usually **fails** on *unbounded* domains (unless extra conditions are imposed). In $\mathbb{R}^n$ find an example of a harmonic function $u(x)$ defined on the unbounded set $\Omega = \{x \in \mathbb{R}^n,\ |x| > 1\}$ which satisfies

$$\triangle u = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

but $u(x)$ is **not** identically equal to zero.

6. (10 points) Consider the Dirichlet problem for Laplace equation

$$\begin{cases} \triangle u(x) = f(x) & \text{in} \quad \Omega \\ u(x) = g(x) & \text{on} \quad \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $f,\ g$ are given smooth functions on Ω and $\partial\Omega$ respectively. Use **Green's identity** to show that a smooth solution $u(x)$ to the above problem, if exists, must be unique. Do the same for the Neumann problem

$$\begin{cases} \triangle u(x) = f(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu}(x) = g(x) & \text{on} \quad \partial\Omega \end{cases}$$

and show that the solution is unique, if exists, up to the addition of a constant.

7. (10 points) Assume f and g are two harmonic functions on domain $\Omega \subset \mathbb{R}^n$ with $f > 0$ everywhere. Show that at a local maximum or minimum point $p \in \Omega$ of $\frac{g}{f}$ (i.e., $D\left(\frac{g}{f}\right) = 0$ at p), we also have

$$\triangle\left(\frac{g}{f}\right) = 0 \quad \text{at} \quad p.$$