Answer all 7 questions. Total 105 points.

QUALIFYING EXAMINATION: REAL ANALYSIS

(15 *pts*) 1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Let g be a nonnegative measurable function on X. Set

$$\nu(E) = \int_E g \, d\mu.$$

Show that ν is a measure on $\mathcal{M}$.

(b) Let f be a nonnegative measurable function on X.
Prove that

$$\int f \, d\mu = \int fg \, d\nu.$$

(c) State the Radon-Nikodym theorem.

(15 *pts*) 2. Let $(X, \mathcal{M}, \mu)$ be a positive measure space with $\mu(X) < \infty$, and let f and g be real-valued measurable functions with

$$\int_X f \, d\mu = \int_X g \, d\mu.$$

Prove that either $f = g$ a.e. or there exists $E \in \mathcal{M}$ such that

$$\int_E f \, d\mu > \int_E g \, d\mu.$$

(15 *pts*) 3. Let $\{f_n\}$ be a sequence of absolutely continuous functions on $[0,1]$, and $f_n(0) = 0$ for $n = 1, 2, \cdots$. Suppose that $\{f_n'\}$ is a Cauchy sequence of $L^1[0,1]$.
Prove that there exists an absolutely continuous function f on $[0,1]$ such that $f_n \to f$ uniformly on $[0,1]$.

(15 *pts*) 4. (a) With Lebesgue measure on $[0,1]$, prove that the closed unit ball

$$\{f \in C[0,1] \,,\, \|f\|_\infty \leq 1\}$$

is *not* compact in $L^1[0,1]$.

(b) Describe all compact subsets of $C[0,1]$.

(15 *pts*) 5. (a) Suppose $f : [a,b] \to \mathbb{R}$ is bounded.
Prove that f is Riemann integrable on $[a,b]$ if and only if f is continuous almost everywhere on $[a,b]$.

(b) Suppose $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable.
Prove that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) cos nx \, dx = 0.$$

(15 *pts*) 6. Let $1 \leq p < q \leq \infty$.

(a) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$.
Prove that $L^q(\mu)$ is contained in $L^p(\mu)$.

(b) Prove that l^p is properly contained in l^q.

(15 *pts*) 7. (a) Find a representation for the bounded linear functionals on $L^p(\mu)$ with $1 \leq p < \infty$ and μ a σ-finite measure. Explain your reason.

(b) Find a representation for the bounded linear functionals on l^p with $1 \leq p < \infty$.