博士班資格考 圖論

Qualifying Examination in Graph Theory 94.2.23

1. (10 points) Prove that if G is a planar graph with n vertices, $n+k$ edges, then G has a cycle of length at most $2(n+k)/(k+2)$.

2. (10 points) Let G be a $2k$-connected graph. Suppose $e_1, e_2, \cdots, e_k$ are vertex disjoint edges of G and v is a vertex of G. Prove that G has k cycles $C_1, C_2, \cdots, C_k$ such that C_i contains v and e_i, and moreover for $i \neq j$, C_i and C_j are vertex disjoint except that they both contain v.

3. (16 points) Prove that an integer sequence $(d_1, d_2, \cdots, d_n)$ is the out-degree sequence of a tournament if and only if $\sum_{i=1}^{n} d_i = \binom{n}{2}$ and for each subset $I \subseteq [n] = \{1, 2, \cdots, n\}$, $\sum_{i \in I} d_i \geq \binom{|I|}{2}$.

4. (16 points) (a) Suppose G is a cubic graph which has a Hamilton cycle. Prove that $\chi'(G) = 3$.

(b) If G is a cubic graph with $\chi'(G) = 3$, then each edge of G is contained in a cycle.

5. (16 points) Prove that if G is an n-vertex planar graph and C is a Hamilton cycle of G, then

$$\sum_{i=1}^{n} (i-2)(\phi'_i - \phi''_i) = 0,$$

where ϕ'_i is the number of faces of length i contained in the interior of C, and ϕ''_i is the number of faces of length i contained in the exterior of C.

6. (16 points) A homomorphism of a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. Suppose there is a homomorphism of an n-vertex graph G to the odd cycle C_{2k+1}. Prove that G has an independent set X with $|X| \geq (\frac{1}{2} - \frac{1}{4k+2})n$.

7. (16 points) Suppose $n \geq 2$. For any $k \geq 0$, find a graph G on $n+k$ vertices such that the automorphism group $Aut(G)$ of G is equal to the symmetric group S_n.